# Does there exist an odd perfect number?

## A simply stated open problem in mathematics

K. N. Raghavan http://www.imsc.res.in/~knr/

A number in this talk will mean a positive whole number: 1, 2, 3, 4, ...

Finding the number of factors: Given a number $n$, can we find how many factors it has? E.g. 10 has 4 factors: 1, 2, 5, 10
16 has 5 factors: 1, 2, 4, 8, 16 6 has 4 factors: 1, 2, 3, 6
If we are given (or otherwise know) the prime factorization of $n$, then we can easily find the number of factors. Illustration 1 $n=100$. $100 = 2^2 \times 5^2$. Any factor of 100 has prime factorization of the form $2^a \times 5^b$ with $a \leq 2$ & $b \leq 2$. So the possible values of $a$ are: 0, 1, 2. So also for $b$. Since $a$ & $b$ can independently have 3 values each, the no. of factors is $3 \times 3 = 9$.
Illustration 2 $2^{11} \times 17^{15} \times 19$ may seem large, but we can nevertheless argue as above to conclude that it has $12 \times 16 \times 2 = 384$ factors.

Finding the sum of the factors: Can we do this? E.g. for 10 we get $1+2+5+10 = 18$. Once again it turns out that if we are given (or somehow know) the prime factorization of $n$, we can easily find the sum of the factors of $n$. Illustration 1 $n=100$. The sum of the nine factors is $2^0 5^0 + 2^0 5^1 + 2^0 5^2 + 2^1 5^0 + 2^1 5^1 + 2^1 5^2 + 2^2 5^0 + 2^2 5^1 + 2^2 5^2$
$= (2^0 + 2^1 + 2^2)(5^0 + 5^1 + 5^2) = 7 \times 31 = 217$.
Illustration 2 The sum of the factors of $2^{11} \times 17^{15} \times 19$ is by a similar reasoning $(2^0 + 2^1 + \ldots + 2^{11})(17^0 + 17^1 + \ldots + 17^{15})(19^0 + 19^1)$.
Using the formula $r^0 + r^1 + \ldots + r^n = \frac{r^{n+1} - 1}{r - 1}$, we obtain the following as answer: $\frac{2^{12} - 1}{2 - 1} \times \frac{17^{16} - 1}{17 - 1} \times 20$

**Perfect numbers** A number $n$ is said to be perfect if the sum of its factors is $2n$.

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | ... | 28 | 29 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Sum of the factors of $n$ | 3 | 4 | 7 | 6 | 12 | 8 | 15 | 13 | 18 | 12 | 28 | 14 | 24 | 24 | 31 | ... | 56 | 30 | ... |

As we can easily check, only 6 & 28 among numbers less than 30 are perfect.

**Theorem:** (Euler 1707-1783) Suppose that $p$ is a prime number such that $2^p-1$ is prime. Then $2^{p-1}(2^p-1)$ is a perfect number. Moreover, every even perfect number arises this way.

**Proof:** see appendix (page 5)

We can try to make a list of the even perfect numbers using Euler's theorem. See the list on the next page.

A prime of the form $2^p-1$ where $p$ is also prime is called a Mersenne prime. By Euler's theorem above, there is a one-to-one correspondence between Mersenne primes and even perfect numbers: given a Mersenne prime $2^p-1$ the corresponding even perfect number is $2^{p-1}(2^p-1)$.

While the table on the next page may lead us to think that there are plenty of Mersenne primes, to date only 49 are known. The largest known one, discovered in Jan 2016, ~~has~~ is $2^{74,207,281}-1$ which has over 22 million digits!

Some open problems: ① Does there exist an odd perfect number?
② Is the number of even perfect numbers finite or infinite?
③ Is $2^p-1$ composite (meaning non prime) for infinitely many primes $p$?

(n) List of even perfect numbers. (See wikipedia page on perfect numbers)

| $p$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $2^p-1$ | 3 | 7 | 31 | 127 | $2047 = 23\times 89$ | 8191 | 131071 | | | | |
| Is $2^p-1$ prime? | ✓ | ✓ | ✓ | ✓ | × | ✓ | ✓ | ✓ | × | × | ✓ |
| $2^{p-1}(2^p-1)$ if perfect | $2\times 3 = 6$ | $2^2\times 7 = 28$ | $2^4\times 31 = 496$ | $2^6\times 127 = 8128$ | × | $2^{12}\times 8191 = 33550336$ | $2^{16}\times 131071$ | ✓ | × | × | ✓ |

To date (Jan 2016) only 49 Mersenne primes (and so also even perfect numbers) are known. For more information refer to the wikipedia pages.

After 31, the next four values of $p$ for which $2^p-1$ is prime are 61, 89, 107, & 127.

**Primality Testing**: Note that $2^p - 1$ grows quickly with $p$:
$2^{107} - 1$ has 33 digits, $2^{521} - 1$ has 157 digits, $2^{1279} - 1$ has 386 digits
Thus in trying to compile the (above) list of even perfect numbers, the following fundamental question arises: given a big number, can we efficiently determine whether or not it is prime?

AGRAWAL-KAYAL-SAXENA (AKS) primality test (2002)
Yes, we can tell easily whether (or not) a number is prime.
2006 Gödel Prize, 2006 Fulkerson Prize awarded to AKS

**Appendix**: (I) Proof of the formula $r^0 + r^1 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}$ $(r \neq 0)$

Put $S = r^0 + r^1 + \dots + r^{n-1} + r^n$
Multiply by $r$: $rS = \quad r^1 + \dots + r^{n-1} + r^n + r^{n+1}$
Subtract: $(1-r)S = r^0 - r^{n+1}$

Thus $S = \frac{r^0 - r^{n+1}}{1-r} = \frac{1 - r^{n+1}}{1-r} = \frac{r^{n+1} - 1}{r-1}$. DONE

(II) If $2^k - 1$ is prime (for a number $k$) then $k$ is prime

Proof ~~by contradiction~~ (of the contrapositive): Suppose not.
Then $k = mn$ with both $m, n \geq 2$. We have $2^k - 1 = 2^{mn} - 1$
Put $x = 2^m$. From the sum of the geometric series above, we have $\frac{x^n - 1}{x - 1} = 1 + \dots + x^{n-1}$
or $(x^n - 1) = (x-1)(1 + \dots + x^{n-1})$
Substituting $2^m$ for $x$: $(2^{mn} - 1) = (2^m - 1)(1 + 2^m + \dots + 2^{m(n-1)})$
Thus $2^k - 1 = (2^{mn} - 1)$ is not prime. DONE

Appendix: (III) Proof of Euler's theorem

Notation: $\sigma(k) :=$ sum of the factors of the number $k$

First part: Suppose that $2^p - 1$ is prime. Write $n = \underbrace{2^{p-1}(2^p-1)}_{\text{prime factorization}}$

Thus $\sigma(n) = (2^0 + \dots + 2^{p-1})(1 + (2^p - 1))$

$= (2^p - 1)\, 2^p$ (where we have used the formula in (I) above)

$= 2n$.

So $n$ is perfect. DONE

Second part: Let $n$ be even perfect. Write $n = 2^{l-1} m$ — (1)

with $l \geq 2$, $m$ odd. On the one hand, we have (easy to see)

$\sigma(n) = (2^0 + \dots + 2^{l-1})\, \sigma(m) = (2^l - 1)\, \sigma(m)$ — (2)

On the other, $\sigma(n) = 2n = 2^l m$ — (3)

From (2) and (3): $(2^l - 1)\, \sigma(m) = 2^l m$ — (4)

Observe that $2^l - 1$, being odd, divides $m$

Write $m = (2^l - 1)\, m'$ — (5)

Substituting (5) into the rhs of (4), we get

$(2^l - 1)\, \sigma(m) = 2^l (2^l - 1)\, m'$

or $\sigma(m) = 2^l m'$ — (6)

If $m' \neq 1$, then by (5), $\sigma(m) \geq 1 + m' + m = 1 + m' + (2^l - 1)\, m'$

$= 1 + 2^l m'$

But this contradicts (6).

So $m' = 1$. And by (5), $m = 2^l - 1$ — (7)

By (6) $\sigma(m) = 2^l = m + 1$

which shows $m$ is prime.

Now by (II) above, we conclude $l$ is prime, say $l = p$.

From (1) & (7), we get $n = 2^{p-1}(2^p - 1)$.

DONE

Some names that a higher secondary student in India wanting to pursue an undergraduate degree in mathematics should be familiar with: Search the internet for more info.

KVPY Kishore Vaigyanik Protsahan Yojana

CMI Chennai Mathematical Institute

ISI Indian Statistical Institute
B. Stat. program at Kolkata
B. Math. program at Bengaluru

IISER Thiruvananthapuram, Mohali (Punjab), Pune, Bhopal, Kolkata, Tirupati

NISER Bhubhaneswar

IISc B.S. (by research) program

Integrated Masters programs: IITK, other IITs, University of Hyderabad