Exercises in Functional Analysis-II

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1. Let c denotes Banach space of all convergent sequences. Show that c^* is isomorphic to l_1. This shows both c_0 (the subspace of sequences converging to 0) and c have isomorphic duals.

2. Prove that a linear functional on a normed linear space is unbounded if and only if its kernel is a proper dense subspace.

3. X is a Banach space. Show that a linear operator T is norm continuous if and only if it is weakly continuous.

4. Define $T : l_1 \mapsto c_0$ by
$$T(\{x_n\}) = \{\sum_{k=n}^{\infty} x_k\}, \ \forall \{x_n\} \in l_1.$$
Show that $T \in B(l_1, c_0))$ and compute T^*.

5. Check the weak and weak* continuity of the map $l^1 \ni (x_n)_{n=1}^{\infty} \mapsto \sum_{n=1}^{\infty} x_n$.

6. Let X be separable Banach space. Prove that the unit ball in X^* is metrizable with respect to the weak* topology. (Hint: Consider $d(\phi, \varphi) = \sum_{n=1}^{\infty} 2^{-n} |\phi(x_n) - \varphi(x_n)$ for some dense $\{x_n\}_{n=1}^{\infty} \in X$.) Show that X^* is separable with respect to the weak* topology.

7. Prove that any infinite-dimensional normed space has a discontinuous linear functional defined on it.

8. Let $X = l^{\infty}$. Define $\varphi_m \in X^*$ by $\varphi_m(\{x_n\}_{n=1}^{\infty}) = x_m$ (the evaluation map at the $m-$th co-ordinate). Show that φ_m does not have a *weak** convergence subsequence, despite the fact that $(X^*)_1$ is compact. This means $(X^*)_1$ is not metrizible.

9. Show that there exists a linear functional φ on l_∞ satisfying
$$\liminf x_n \leq \varphi(x) \leq \limsup x_n \ \ \forall x = \{x_n\}.$$

10. Show that l_1 is not reflexive. (Use Hahn-Banach theorem.)

11. Show that every infinite orthonormal sequence converges weakly to 0.

12. Let H be a Hilbert space and $, E \subseteq H$ be an orthonormal basis for H. Show that a sequence $\{x_n\} \subseteq H$ converges weakly to 0 if and only if $\sup\{\|x_n\| : n \geq 1\} < \infty$ and $\langle x_n, e \rangle \longmapsto 0$ for all $e \in E$

13. Let H be a separable Hilbert space and $\{e_n\}$ be an ONB for H. Define $T_n, S_n \in B(H)$ for $n \geq 1$ by

$$T_n(\xi) = \langle \xi, e_n \rangle e_1; \quad S_n(\xi) = \langle \xi, e_1 \rangle e_n \ \forall \xi \in H.$$

Show that T_n converges strongly but not in norm, and S_n converges in weak operator topology but not strongly.