**Exercises in Functional Analysis**

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(Baires theorem, Uniform boundedness principle, open mapping theorem, closed graph theorem.)

1. Show that a perfect subset of a complete metric space is uncountable. (We say a point $x$ is an accumulation point for $S$, if for every $r > 0$, $B_r(x) \cap S$ contains a point other than $x$. A set is perfect if it equals to all its accumulation points.)

2. Show that any infinite dimensional Banach space does not admit a countable Hamel basis. (Hamel Basis is the usual basis, a linear independent set whose linear span is whole Banach space.)

3. Show that the subspace of piecewise-linear functions is dense in $C([0,1])$. (Hint: Use the uniform continuity of continuous functions on $[0,1]$ and total boundedness of $[0,1]$.)

4. Let $\mathcal{D} = \{f \in C([0,1]) : \text{there exists } x \in [0,1] \text{ such that } f \text{ is differentiable at } x\}$, and for $n, m \in \mathbb{N}$, let $A_{n,m} =$

$$\left\{ f \in C([0,1]) : \text{there exists } x \in (0,1) \text{ such that } \frac{|f(x) - f(t)|}{|x - t|} \leq n \text{ if } |x - t| < \frac{1}{m} \right\}.$$

   Show that each $A_{n,m}$ is closed and $\mathcal{D} \subseteq \bigcup_{n,m \in \mathbb{N}} A_{n,m}$. (Hint: Bolzono-Weierstrass Theorem will be useful.)

5. Let $X$ be a complete metric space. Let $f_n : X \mapsto \mathbb{C}$ are continuous for all $n \in \mathbb{N}$ such that $f_n(x) \mapsto f(x)$ as $n \mapsto \infty$ for all $n \in \mathbb{N}$. Show that the set of points where $f$ is discontinuous is meager in $X$. In particular the set of points where $f$ is continuous is dense on $X$.

   (Hint: Step 1: Define
$$osc(f)(x) = \inf_{r>0} w(f)(r, x)$$
   where $w(f)(r, x) = \sup_{y,z \in B_r(x)} |f(z) - f(y)|$. Show that $f$ is continuous at $x$ if and only if $osc(f)(x) = 0$.

   Step 2: Show that for any given open ball $B$ and $\epsilon > 0$, there exists a ball $B_0 \subseteq B$ and $m \in \mathbb{N}$ such that $|f_m(x) - f(x)| \leq \epsilon \ \forall x \in B_0$.

   Step 3: Define $F_n = \{x \in X : osc(f)(x) \geq \frac{1}{n}\}$. )

6. Prove the following version of uniform boundedness principle: If a family of bounded linear operators $\{T_\lambda\}_{\lambda \in \Lambda}$ on a Banach space $X$ do not have a uniform bound, that is $sup_{\lambda \in \Lambda} \|T_\lambda\| = \infty$, then show that the set
$$S = \{x \in X : sup_{\lambda \in \Lambda} \|T_\lambda x\| = \infty\}$$
is dense in $X$.

7. Let $X$ be Banach space and $S \subseteq X$ weakly bounded (i.e. $\sup\{|\varphi(s)| : s \in S\} < \infty$ for all $\varphi \in X^*$). Show that $S$ is bounded in norm.

8. Let $X$ be a Banach space and $\{\varphi_n\}_{n=1}^{\infty} \subseteq X^*$ be a sequence such that $\sum_{n=1}^{\infty} \varphi_n(x)$ converges for every $x \in X$. Show that $\sum_{n=1}^{\infty} \frac{\|\varphi_n\|}{2^n}$ is convergent.

9. Let $x = \{x_n\}_{n=1}^{\infty}$ be a sequence of complex number such that the series $\sum_{n=1}^{\infty} x_n y_n$ is convergent for all $y = \{y_n\}_{n=1}^{\infty} \in c_0$. Prove that $x \in l^1$.

10. Suppose that $X$ and $Y$ are Banach spaces and that $B : X \times Y \mapsto \mathbb{C}$ is a separately continuous bilinear mapping (that is $B(x, \cdot) : Y \mapsto \mathbb{C}$ is continuous for each $x \in X$ and $B(\cdot, y) : X \mapsto \mathbb{C}$ is continuous for each $y \in Y$). Then prove that $B$ is jointly continuous.

11. Suppose that $X$ forms a Banach space with respect to two norms $\|.\|_1$ and $\|.\|_2$, and that there exists a positive constant $C$ such that $\|x\|_1 \le C\|x\|_2$ for all $x \in X$. Then prove that there exists a positive constant $D$ such that $\|x\|_2 \le D\|x\|_1$ for all $x \in X$. Conclude that if $X$ is a Banach space with respect to two different norms then they are either equivalent or non-comparable (i.e., neither is coarser than the other).

12. Let a Banach space $X = X_1 \oplus X_2$, where the direct sum is algebraic (i.e. $X_1$ and $X_2$ are closed subspaces such that $X_1 \cap X_2 = \emptyset$ and $X_1 + X_2 = X$). Sow that any $x \in X$ has a unique decomposition $x = x_1 + x_2$ and that there exists a $c > 0$ such that $\|x_1\| + \|x_2\| \le c\|x\|$.

13. Let $C^1([0,1]) \subseteq C([0,1])$ the subspace of continuously differentiable functions. Define $T : C^1([0,1]) \mapsto C([0,1])$ by
$$Tf = f'.$$
Show that $T$ is unbounded but still has closed graph.

14. Let $X$ be a Banach space and $P : X \mapsto X$ is a linear map satisfying $P^2 = P$. Further if $Range(P)$ and $Ker(P)$ are closed, then show that $P$ is continuous.

15. Let $X, Y$ be Banach spaces and $T : X \mapsto Y$ a linear map. If there exists a linear map $S : Y^* \mapsto X^*$ satisfying
$$(S\varphi)(x) = \varphi(Tx) \;\; \forall \varphi \in Y^*, x \in X,$$
then prove that $T$ is bounded.