(1) Throughout this course, by a ring we shall mean a commutative ring with identity 1, such that $1 \neq 0$.

(2) An integral domain is a ring A such that if $x, y \in A$ and $xy = 0$, then either $x = 0$ or $y = 0$.

(3) An ideal $\mathfrak{p} \subset A$ is called prime if

(a) $\mathfrak{p} \subsetneq A$, and

(b) $x, y \notin \mathfrak{p} \Rightarrow xy \notin \mathfrak{p}$

or equivalently, $A/\mathfrak{p}$ is a non-zero integral domain.

(4) An ideal $\mathfrak{m} \subset A$ is called maximal if

(a) $\mathfrak{m} \subsetneq A$, and

(b) if $x \in A \setminus \mathfrak{m}$, then $\mathfrak{m} + (x) = A$

or equivalently, $A/\mathfrak{m}$ is a non-zero field.

(5) The first important result we want to prove is that every ring has maximal ideals. The idea of the proof is the following, start with the 0 ideal and keep adding elements until we get a maximal ideal. The formal proof follows.

<u>Definition</u>: A subset $S \subset A$ is called multiplicatively closed if $x, y \in S \Rightarrow xy \in S$, $1 \in S$ and $0 \notin S$.

<u>Lemma</u>: Let S be a multiplicative set. Let Σ be the collection of ideals of A such that $I \cap S = \phi$. There is a partial ordering by inclusion on Σ. Then Σ has maximal elements and every maximal element is a prime ideal of A.

Pf: Observe that Σ is non-empty as $0 \in \Sigma$. That Σ has maximal elements follows from Zorn's lemma. Let Λ be a totally ordered set, that is, for $\alpha, \beta \in \Lambda$, either $\alpha \leq \beta$ or $\beta \leq \alpha$. Suppose for each $\alpha \in \Lambda$, we are given $I_\alpha \in \Sigma$ such that if $\alpha \leq \beta$, then $I_\alpha \subset I_\beta$. Consider the ideal $J = \bigcup_{\alpha \in \Lambda} I_\alpha$.

$J \in \Sigma$ as $J \cap S = \left(\bigcup_{\alpha \in \Lambda} I_\alpha\right) \cap S = \bigcup_{\alpha \in \Lambda} I_\alpha \cap S = \phi$.

Moreover, J is an upper bound for the chain $\{I_\alpha\}_{\alpha \in \Lambda}$. Thus, by Zorn's lemma, Σ has maximal elements. Let $\mathfrak{p} \in \Sigma$ be a maximal element. We claim the $\mathfrak{p}$ is prime. If not, $\exists\ x, y \notin \mathfrak{p}$ but $xy \in \mathfrak{p}$. Since $\mathfrak{p} \subsetneq \mathfrak{p} + (x), \mathfrak{p} + (y) \Rightarrow (\mathfrak{p} + (x)) \cap S \neq \phi \Rightarrow$ $\exists\ a + bx \in S$ and similarly, $a' + b'y \in S \Rightarrow (a+bx)(a'+b'y) \in S$. But as $xy \in \mathfrak{p}$, this element is also in $\mathfrak{p}$, $\Rightarrow \mathfrak{p} \cap S \neq \phi$, contradicting $\mathfrak{p} \in \Sigma$. This completes the proof of the lemma.

Thm: Let A be a ring in which $1 \neq 0$. Then A has maximal ideals.

Pf: Apply the above lemma by taking $S = \{1\}$. Let m be a maximal element such that $m \cap S = \phi$. Then if $x \in A \setminus m$, $(m + (x)) \cap S \neq \phi$, $\Rightarrow m + (x) = A$.

Cor: If $x \in A$ is not a unit, then there is a maximal ideal containing x.

Pf: Let S be the set of units in A. Then S is a multiplicative set. Let Σ be the collection of ideals I such that $x \in I$ and $I \cap S = \phi$. The ideal $(x) \in \Sigma$ and so $\Sigma \neq \phi$. Let m be a maximal element of Σ. We claim m is maximal, and we omit the simple proof.

UPSHOT: Every ring has prime ideals.

⑥ Let $f: A \to B$ be a ring homomorphism. If $q \subset B$ is a prime ideal, then $f^{-1}(q) \subset A$ is a prime ideal. To see this, note that the natural map $\bar{f}$ is an inclusion. Since B/q is a domain, $\Rightarrow A/f^{-1}(q)$ is a domain, $\Rightarrow f^{-1}(q)$ is a prime ideal.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A/f^{-1}(q) \arrow[r, hook, "\bar{f}"'] & B/q \end{tikzcd}$$

REMARK: The contraction of a maximal ideal need not be a maximal ideal. Consider the inclusion $\mathbb{Z} \xhookrightarrow{i} \mathbb{Q}$. Then (0) is maximal in $\mathbb{Q}$ and $i^{-1}(0) = (0)$ is not maximal in $\mathbb{Z}$.

⑦ Let A be a ring and let $I \subsetneq A$ be an ideal. Let $\Phi: A \to A/I$ be the natural map. There is 1-1 correspondence between the prime ideals of A/I and the prime ideals of A which contain I, explicitly given by

$$A \xrightarrow{\Phi} A/I$$
$$\Phi^{-1}(p) \longleftarrow p$$

We remark that under this correspondence, maximal ideals DO contract to maximal ideals. This is because if $m \subset A/I$ is maximal, then $A/\Phi^{-1}(m) \hookrightarrow (A/I)/m$ and this is also a surjection. Thus, it is an isomorphism. Since $(A/I)/m$ is a field $\Rightarrow A/\Phi^{-1}(m)$ is a field, $\Rightarrow \Phi^{-1}(m)$ is maximal.

⑧ Nilradical : Define $\mathrm{Nil}(A) = \{ x \in A \mid \exists\, n > 0 \text{ such that } x^n = 0 \}$

Propn : $\mathrm{Nil}(A) = \bigcap_{\text{primes}} \mathfrak{p}$

Pf : It is clear that $\mathrm{Nil}(A) \subset \bigcap_{\text{primes}} \mathfrak{p}$, since if $x \in \mathrm{Nil}(A)$,

$\Rightarrow \exists\, n > 0$ such that $x^n = 0 \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$.

Suppose $x \notin \mathrm{Nil}(A)$. Let $S = \{1, x, x^2, \ldots\}$, then $0 \notin S$ and S is multiplicative. Let Σ be the collection of ideals which do not meet S. Then $(0) \in \Sigma$ and so $\Sigma \neq \phi$, and so there is a maximal element in Σ which is prime, say $\mathfrak{p}$. $\Rightarrow$ $\mathfrak{p} \cap S = \phi$, $\Rightarrow x \notin \mathfrak{p}$, Thus $x \notin \bigcap_{\text{primes}} \mathfrak{p}$. This completes the proof.

⑨ Jacobson Radical : Define $\mathrm{Jac}(A) = \bigcap_{\substack{\text{maximal}\\ \text{ideals}}} \mathfrak{m}$.

Propn : $x \in \mathrm{Jac}(A) \iff 1 - xy$ is a unit in A $\forall\, y \in A$.

Pf : If $x \in \mathrm{Jac}(A)$ and $y \in A$, then $xy \in \mathfrak{m}$ for every maximal ideal, $\Rightarrow 1 - xy \notin \mathfrak{m}$ for every maximal ideal, or else the maximal ideal would contain 1. $\Rightarrow 1 - xy$ is a unit.

$$\Rightarrow \mathrm{Jac}(A) \subset \{ x \in A \mid 1 - xy \text{ is a unit } \forall\, y \in A \}$$

Now suppose $x \notin \mathrm{Jac}(A)$, $\Rightarrow \exists$ maximal ideal $\mathfrak{m}$ such that $x \notin \mathfrak{m}$, $\Rightarrow \mathfrak{m} + (x) = A$, $\Rightarrow 1 = a + xy \Rightarrow a = 1 - xy$, $\Rightarrow \exists\, y$ such that $1 - xy$ is not a unit.

This completes the proof.

Chapter 2

① <u>Nakayama's lemma</u>: Let M be a finitely generated A-module, and assume that $M = JM$ for some ideal $J \subset \mathrm{Jac}(A)$. Then $M = 0$.

<u>Pf</u>: Assume that $m_1, m_2, \ldots m_n$ are generators for M as an A-module. Since $M = JM$, $m_i = \sum_{j=1}^{n} a_{ji} m_j$ for $a_{ji} \in J$. If $A = (a_{ij})$, then we may write the above equations in the form $(A - I)\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0$.

Multiplying by the adjoint of $A - I$ on the left, we get $\det(A - I)\, m_i = 0 \ \forall i$. But note $\det(A - I)$ is an element of the form $\pm(1 + a_1 + a_2 + \cdots + a_n)$ where $a_i \in J^i$. In particular, it is a unit. Thus, each $m_i = 0$, $\Rightarrow M = 0$.

② <u>Cayley-Hamilton theorem</u>: Let M be a finitely generated A-mod and suppose $\phi \in \mathrm{End}_A(M)$ is such that $\phi(M) \subset JM$. Then ϕ satisfies a polynomial equation of the type $\phi^n + a_1 \phi^{n-1} + \cdots + a_n = 0$ where $a_i \in J^i$.

<u>Pf</u>: The ideal is to modify the proof of Nakayama's lemma slightly. Let $R \subset \mathrm{End}_A(M)$ be the commutative subring generated by $\langle A, I, \phi \rangle$. Then M is an R-module in a natural way. Moreover, M is finitely generated as an R-module, since it is finitely generated as an A-module. Let $m_1, m_2, \ldots m_n$ be its generators. Then $\phi(m_i) = \sum_{j=1}^{n} a_{ji} m_j$, which we rewrite as

$(A - \phi)\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}$. Now $A - \phi$ is a matrix with coefficients in R.

Taking adjoint, we get $\det(A - \phi)\, m_i = 0 \ \forall i = 1, 2, \ldots, n$.

But $\det(A - \phi) = \pm(\phi^n + a_1 \phi^{n-1} + \cdots + a_n)$ $a_i \in I^i$. This completes the proof of the Cayley-Hamilton theorem.

(3) <u>Tensor Products</u> : Let M and N be A-modules. We want to define a pair $(\varphi, M \otimes_A N)$ consisting of

- An A-module which we denote $M \otimes_A N$
- An A-bilinear map $\varphi : M \times N \longrightarrow M \otimes_A N$

which is universal in the following sense

- For any A-module P and an A-bilinear map $f : M \times N \longrightarrow P$, this factors uniquely as follows

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[dr, "\varphi"'] & P \\ & M \otimes_A N \arrow[u, dashed] \end{tikzcd} \quad \exists!\ \tilde{f} \rightarrow A\text{-linear}$$

<u>Remark</u> : The above is an example of a universal property. Here is a simpler example of a universal property. Let G be a group and let $K \subset G$ be a normal subgroup. Consider the pair $(\pi, G/K)$, where $\pi : G \longrightarrow G/K$ is the natural map. This pair is universal among all pairs (f, H) where $f : G \to H$ is a group homomorphism such that $K \subset \operatorname{Ker} f$

In other words, any homomorphism $f : G \to H$ such that $K \subset \operatorname{Ker} f$ factors as

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "\pi"'] & H \\ & G/K \arrow[u, dashed] \end{tikzcd} \quad \exists!\ \tilde{f}$$

- Let us denote by $\operatorname{Bil}_A(M \times N, P)$ the space of all A-bilinear maps $M \times N \longrightarrow P$, and by $\operatorname{Hom}_A(M, N)$ the space of A-linear maps $M \to N$. Both these have natural A-module structures and there is a natural isomorphism $\operatorname{Hom}_A(M, \operatorname{Hom}_A(N, P)) \longrightarrow \operatorname{Bil}_A(M \times N, P)$
  given by $\varphi : M \to \operatorname{Hom}_A(N, P) \longmapsto (m, n) \mapsto \varphi(m)(n)$

  The universal property of tensor products is equivalent to saying, there is an isomorphism $\operatorname{Bil}_A(M \times N, P) \longrightarrow \operatorname{Hom}_A(M \otimes_A N, P)$.

Thus, we have natural isomorphisms

$$\mathrm{Hom}_A(M, \mathrm{Hom}_A(N,P)) \xrightarrow{\sim} \mathrm{Bil}_A(M\times N, P) \xrightarrow{\sim} \mathrm{Hom}_A(M\otimes_A N, P).$$

<u>Existence of the pair $(\varphi, M\otimes_A N)$ having the required universal property.</u>

We use similar notation as in Atiyah & MacDonald.

Consider the free A-module with basis $(m,n)\in M\times N$. In other words, we mean the free module $C = \bigoplus_{(m,n)\in M\times N} A\cdot(m,n)$.

Let $D\subset C$ be the submodule generated by elements of the form

- $a(m,n) - (am,n)$
- $a(m,n) - (m,an)$
- $(m_1+m_2,n) - (m_1,n) - (m_2,n)$
- $(m,n_1+n_2) - (m,n_1) - (m,n_2)$

Then there is a natural map $C \xrightarrow{\pi} C/D$. Further, consider the set map $M\times N \to C$ given by $(m,n)\mapsto (m,n)$ ($\to$ this denotes the basis vector)

Call the composite map $M\times N \xrightarrow{\varphi} C/D$.

<u>Claim 1</u>: φ is A-bilinear.

This is straightforward check which follows from the definition of D.

<u>Claim 2</u>: φ is universal among such bilinear maps.

Suppose $f: M\times N \to P$ is an A-bilinear map. Then we want to define an A-linear map $\tilde{f}: C/D \to P$ which makes the following diagram commute

$$\begin{tikzcd} M\times N \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & P \\ & C/D \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

$\tilde{f} \to A$-linear

First let us define a map $C\to P$. Since C is a free A-module, to define an A-linear hom $C\to P$, it is enough to specify where the basis elements go. Define a map by sending the basis element $(m,n) \longmapsto f(m,n)$. We extend this hom to the whole of C A-linearly. Next we check that this hom is 0 on D. It is enough to check this is 0 on the generators of D. For example, $(m_1+m_2, n) - (m_1, n) - (m_2, n) \mapsto$

$$f(m_1+m_2, n) - f(m_1, n) - f(m_2, n) = 0 \text{ as } f \text{ is } A\text{-bilinear}.$$

Similarly, $a(m,n) - (am, n) \longmapsto a f(m,n) - f(am, n) = 0$.

The other checks are equally simple. Thus, we get a commutative diagram

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "\varphi"'] & P \\ C/D \arrow[ur, "\tilde{f}"'] & \end{tikzcd} \quad \tilde{f} \to A\text{-linear}.$$

Moreover, for any $\tilde{f}$ which makes the above diagram commute, it is forced that $\tilde{f}(\varphi(m,n)) = f(m,n)$. Since as an A-module C/D is generated by $\varphi(m,n)$, we get that $\tilde{f}$ is unique. For two modules M and N, we shall denote the map $M \times N \xrightarrow{\varphi} C/D$ by $M \times N \longrightarrow M \otimes_A N$, and given by $(m,n) \longmapsto m \otimes_A n$. We remark that, by construction one has

- $(am) \otimes_A n = a(m \otimes_A n) = m \otimes_A (an)$
- $(m_1 + m_2) \otimes_A n = m_1 \otimes_A n + m_2 \otimes_A n$
- $m \otimes_A (n_1 + n_2) = m \otimes_A n_1 + m \otimes_A n_2$

Suppose $f : M \to M'$ and $g : N \to N'$ are A-module homomorphisms. We want to construct an A-module homomorphism $f \otimes g : M \otimes_A M' \longrightarrow N \otimes_A N'$. Recall that we have only one way of constructing A-module homomorphisms from $M \otimes_A M'$, which is to give an A-bilinear map $M \times M' \longrightarrow N \otimes_A N'$. Define such a map by $(m, m') \longmapsto f(m) \otimes_A g(m')$. It is trivial to check this map is A-bilinear, and so it gives an A-linear map $M \otimes_A M' \longrightarrow N \otimes_A N'$, given by $m \otimes_A m' \longmapsto f(m) \otimes_A g(m')$.

Using the same method as above, one can prove the following

- $(M \otimes_A N) \otimes_A P \xrightarrow{\sim} M \otimes_A (N \otimes_A P)$
- $(M \oplus N) \otimes_A P \xrightarrow{\sim} M \otimes_A P \oplus N \otimes_A P$

<u>Propn</u>: There is a natural map $A \otimes_A M \longrightarrow M$ which is an isomorphism.

<u>Pf</u>: Since the map $A \times M \longrightarrow M$ given by $(a, m) \longmapsto am$ is A-bilinear, we get a A-linear map $A \otimes_A M \longrightarrow M$.

This is surjective since $1 \otimes_A m \longmapsto m$. To see this is injective, suppose $\sum_{i=1}^{n} a_i \otimes_A m_i \longmapsto 0$, $\Rightarrow \sum_{i=1}^{n} a_i m_i = 0$ in M.

$$\sum_{i=1}^{n} a_i \otimes_A m_i = \sum_{i=1}^{n} 1 \otimes_A a_i m_i = 1 \otimes_A \left(\sum_{i=}^{n} a_i m_i \right) = 0.$$

$\Rightarrow$ Injective.

④ Right exactness of Tensor Products.

<u>Chain complexes</u>: A complex of A-modules is a sequence

$$\cdots \to M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \longrightarrow \cdots \quad \text{where}$$

- Each M_i is an A-module
- $d_i : M_i \longrightarrow M_{i-1}$ is an A-module hom
- $d_i \circ d_{i+1} = 0$, that is, $\operatorname{Im} d_{i+1} \subset \operatorname{Ker} d_i$

We say the complex is exact at i if $\operatorname{Im} d_{i+1} = \operatorname{Ker} d_i$

We define the homology groups of the complex $M_\bullet$ by

$$H_i(M_\bullet) = \frac{\operatorname{Ker} d_i}{\operatorname{Im} d_{i+1}}$$

<u>Remark</u>: $H_i(M_\bullet) = 0 \iff$ exact at i.

A short exact sequence is an exact complex of the type $0 \to M' \to M \to M'' \to 0$, that is, $M' \subset M$ and $M'' \cong M/M'$.

<u>Propn</u>: Suppose $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact, then the complex $M' \otimes_A N \xrightarrow{f \otimes 1} M \otimes_A N \xrightarrow{g \otimes 1} M'' \otimes_A N \to 0$ is exact.

<u>Pf</u>: The proof is based on the following result

① Let $C' \xrightarrow{h'} C \xrightarrow{h} C'' \to 0$ be a complex of A-modules. Then this complex is exact $\iff$ $\forall$ A-modules P, the resulting complex

$$0 \to \operatorname{Hom}_A(C'', P) \to \operatorname{Hom}_A(C, P) \to \operatorname{Hom}_A(C', P) \text{ is exact.}$$

We leave the proof of this as an exercise. Applying $\operatorname{Hom}_A(\_, \operatorname{Hom}_A(N, P))$ to the exact sequence $M' \to M \to M'' \to 0$, we get the following sequence is exact

$$\begin{array}{ccccc} 0 \to \operatorname{Hom}_A(M'', \operatorname{Hom}_A(N,P)) & \to & \operatorname{Hom}_A(M, \operatorname{Hom}_A(N,P)) & \to & \operatorname{Hom}_A(M', \operatorname{Hom}_A(N,P)) \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ (*) \quad 0 \to \operatorname{Hom}_A(M'' \otimes_A N, P) & \longrightarrow & \operatorname{Hom}_A(M \otimes_A N, P) & \longrightarrow & \operatorname{Hom}_A(M' \otimes_A N, P) \end{array}$$

$\Rightarrow$ $\forall$ P, the sequence $(*)$ is exact $\Rightarrow$ using lemma

$$M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0 \text{ is exact.}$$

<u>Remark</u>: Tensor product is not left exact, as the following example shows. Consider the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

Tensoring this with $\mathbb{Z}/2\mathbb{Z}$ yields the complex

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to 0$$

One of the exercises asks to prove that $A/I \otimes_A M \xrightarrow{\sim} M/IM$.
Using this exercise we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \arrow[r, "2"] \arrow[d, "\wr"'] & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "\wr"'] & \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "\wr"'] & 0 \\
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "2"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

The bottom row is not exact since $\mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/2\mathbb{Z}$ is not an inclusion.

<u>Definition</u>: An A-module M is called flat if for every short exact sequence $0 \to N' \to N \to N'' \to 0$, the resulting complex $0 \to N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M \to 0$ is exact.

We remark that in the above, the only check which needs to be done is that $0 \to N' \otimes_A M \to N \otimes_A M$ is exact.