Additional Problems for Part-2

1. Let A be a measure zero subset of $\mathbb{R}^k$. Prove that $A \times \mathbb{R}^l$ is of measure zero in $\mathbb{R}^{k+l}$.

2. Exhibit a smooth map $f: \mathbb{R} \to \mathbb{R}$ whose set of critical values is dense. (Hint: write the rational numbers as $r_0, r_1, \ldots$ and construct an appropriate function $[i, i+1] \to \mathbb{R}$ which has a critical value at r_i)

3. Prove that the sphere S^k is simply connected if $k > 1$.
(Hint: Apply Sard's theorem to a smooth function $f: S^1 \to S^k$ for $k > 1$.)

4. Suppose that $f_0, f_1 : X \to Y$ are homotopic. Prove that there exists a homotopy $\tilde{F}: X \times I \to Y$ such that $\tilde{F}(x,t) = f_0(x)$ $\forall\, t \in [0, 1/4]$ and $\tilde{F}(x,t) = f_1(x)$ $\forall\, t \in [3/4, 1]$.

5. Prove that every connected manifold is path connected.

6. A manifold is called contractible if the identity map is homotopic to the constant map. Prove that X is contractible $\Leftrightarrow$ $\forall\, Y$, any two maps from Y to X are homotopic.

7. Prove that $\mathbb{R}^k$ is contractible.

8. Prove that the antipodal map $x \mapsto -x$ of $S^k \to S^k$ is homotopic to the identity if k is odd. (Hint: Use homotopies of the type $\begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix}$)

9. Prove that the fixed point in the Brouwer's fixed point theorem need not be an interior point.

10. Find maps of the solid torus into itself having no fixed points. Where does the proof of Brouwer's Theorem fail?

11. Prove that the Brouwer's theorem is false for the open ball $|x|^2 < a$.

Additional Problems for Part-3

1. Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that for almost every $a \in \mathbb{R}^n$ the translate $X+a$ intersects Y transversely.

2. Prove that there exists a complex number z such that
$$z^7 + \cos(|z|^2)(1+93z^4) = 0.$$

3. Prove that intersection theory is vacuous in contractible manifolds: Suppose Y is contractible and $\dim Y > 0$, for every X, Z such that $\dim X + \dim Z = \dim Y$ (X, Z submanifolds), the intersection number of X and Z is 0.

4. Prove that no compact manifold (other than the one-point space) is contractable.

5. Suppose $f: X \longrightarrow S^k$ is smooth, X compact and $0 < \dim X < k$. Then for all $Z \subset S^k$ of complimentary dimension, $I_2(X, Z) = 0$.

6. Prove that S^2 and the torus are not diffeomorphic.

7. Let $\beta = \{v_1, \dots, v_k\}$ be an ordered basis of V. Show that:

a) Replacing one ~~of~~ v_i by a multiple cv_i yields an equivalently oriented basis if $c > 0$ and an oppositely oriented one if $c < 0$.

b) Transposing two elements (i.e. interchanging v_i & v_j) yields an oppositely oriented basis.

c) Subtracting ~~one~~ from one ~~of~~ v_i a linear combination of others yields an equivalently oriented basis.

8. Suppose that V is the direct sum of V_1 and V_2. Prove that the direct sum orientation from ~~on~~ $V_1 \oplus V_2$ equals $(-1)^{\dim V_1 \dim V_2}$ times the orientation ~~on~~ from $V_2 \oplus V_1$.

~~12.~~

9. Suppose that $f : X \to Y$ is a diffeomorphism of connected oriented manifolds with boundary. Show that if $df_x : T_x X \longrightarrow T_{f(x)}(Y)$ preserves orientation at one point x, then f preserves orientation globally.

10. Prove that any compact hypersurface in Euclidean space is orientable.

11. Prove that the Möbius band is not orientable.

12. Let $f(z) = 1/z$ on the circle of radius r in $\mathbb{C}$.
    a) Compute $\deg\left(f/_{|f|}\right)$.
    b) Why does the proof of Fundamental Theorem of Algebra ~~fail in $\mathbb{R}$?~~ not imply that $1/z = 0$ for some $z \in \mathbb{C}$?

13. Prove that a map $f : S^1 \to \mathbb{R}^2 - \{0\}$ extends to the whole ball $B = \{|z| \leq 1\}$ if and only if $\deg(f) = 0$.

Part - 4

1. Let Y be a compact submanifold of $\mathbb{R}^M$ and let $w \in \mathbb{R}^M$. Show that there exists ~~an~~ a point $y \in Y$ closest to w and $w - y$ is normal to Y at y (i.e., $w - y \in N_y Y$).