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Tutorial on Differential Topology

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The following are some of the problems which were discussed in tutorial classes. There are many other problems given in my lecture notes as exercises, which is not discussed in tutorial classes.

1. (i) Show that a finite set has measure zero.

(ii Show that a countable set has measure zero.

2. If $A \subset R^n$ has measure zero in $\mathbb{R}^n$, then $A \times \mathbb{R}^k \subset \mathbb{R}^{n+k}$ has measure zero in $\mathbb{R}^{n+k}$.

3. Show that $\mathbb{R}^k$ is of measure zero in $\mathbb{R}^n$, $k < n$.

4. If U is a non-empty open set in $\mathbb{R}^n$, then U is not of measure zero in $\mathbb{R}^n$.

5. If If M and N are manifolds with $\dim M < \dim N$, then a smooth map $f : M \to N$ cannot be surjective.

6. If $f : M \to N$ is a smooth map with set of critical points C, then the set $N - f(C)$ is dense in N.

7. If $f_i : M \to N$ is a countable family of smooth maps, then the set of common regular values of all the f_i is dense in N.

8. If $f : M \to N$ is a C^1 map, and Z is a set of measure zero in M, then the set $f(Z)$ has measure zero in N.

9. Every manifold is metrizable.

10. If M is a manifold without boundary, and $\pi : M \to \mathbb{R}$ is a smooth map such that 0 is a regular value of π, then $\pi^{-1}([0, \infty))$ is a submanifold of M with boundary $\pi^{-1}(0)$.

11. Show that the space $M_k(m, n, \mathbb{R})$ of real $m \times n$ matrices of rank k, $0 < k \leq \min\ (m, n)$, with the induced topology of the space of the real $m \times n$ matrices $M(m, n, \mathbb{R}) \equiv \mathbb{R}^{mn}$, is a smooth manifold of dimensional $k(m + n - k)$.

12. Let $A : \mathbb{R}^k \to \mathbb{R}^n$ be a linear map, and $V \subset \mathbb{R}^n$ be a vector subspace. True or false :

$$A \pitchfork V \ \text{ means that } A(\mathbb{R}^k) + V = \mathbb{R}^n.$$

13. Let V and W be linear subspaces of R^n.

(i) When is $V \pitchfork W$?

(ii) True or false :
Spaces $V \times \{0\}$ and the diagonal in $V \times V$ intersect transversally.

14. True of false : Subspaces of symmetric $\{A^t = A\}$ and skew-symmetric $\{A^t = -A\}$ matrices in the space of $n \times n$ matrices $M(n)$. intersect transversally.

15. If V_1, V_2, V_3 are linear subspaces of $\mathbb{R}^n$, say that they have "normal intersection" if

$$V_i \,\overline{\pitchfork}\, (V_j \cap V_k) \text{whenever } i \neq j \text{ and } i \neq k.$$

Prove that this holds if and only if

$$\text{Codim } (V_1 \cap V_2 \cap V_3) = \text{Codim } V_1 + \text{Codim } V_2 + \text{Codim } V_3.$$

16. Show that the intersection of two transverse submanifolds Z_1 and Z_2 of a manifold N is a submanifold of Z_1, and

$$\text{Codim } (Z_1 \cap Z_2) = \text{Codim } Z_1 + \text{Codim } Z_2.$$

17. If X and Z are transversal submanifolds of Y, and if $y \in X \cap Z$, then

$$T_y(X \cap Z) = T_y(X) \cap T_y(Z).$$

(The tengent space to the intersection is the intersection of the tangent spaces.)

18. Recall that Euler's identity for a homogeneous polynomial of degree m in k variables $p(x_1, \ldots, x_k)$ is

$$p(tx_1, \ldots, tx_k) = t^m p(x_1, \ldots, x_k)$$

(a) Prove that the set of points x where $p(x) = a$ is a $(k-1)$-dimensional submanifold of $\mathbb{R}^k$, provided $a \neq 0$.

(b) Show that the manifolds obtained for $a > 0$ are all diffeomorphic, as are those with $a < 0$.

19. Let $p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$ be a polynomial with complex coefficients. Consider the map $\mathbb{C} \to \mathbb{C}$ of the complex plane defined by $z \mapsto p(z)$. Show that this map is a submersion except at finitely many points.

20. Show that the set of all 2×2 matrices of rank 1 is a 3-dimensional submanifold of $\mathbb{R}^4 = M(2)$ ($M(2)$ is the set of all 2×2 matrices with real entries).

Note. There are few more which I do not remember now.