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Lecture on Differential Topology Part 3

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1. Transversality

Definition 1.1. Let M and N be manifolds, and Z be a submanifold of N. Then a smooth map $f : M \longrightarrow N$ is called **transverse to Z at a point** $x \in f^{-1}(Z)$ (written $f \overline{\pitchfork}_x Z$) if

$$df_x(\tau(M)_x) + \tau(Z)_{f(x)} = \tau(N)_{f(x)} \text{ (not necessarily a direct sum) .}$$

We say that f is **transverse to** Z (written $f \overline{\pitchfork} Z$) if

$$\text{either (1) } f^{-1}(Z) = \emptyset, \quad \text{or (2) } f \overline{\pitchfork}_x Z \text{ for every } x \in f^{-1}(Z).$$

If $\dim M + \dim Z < \dim N$, then the condition (2) is not possible, so in this case $f \overline{\pitchfork} Z$ means that $f(M)$ does not intersect Z.

The condition (2) is equivalent to saying that df_x induces an epimorphism $\overline{df_x} = \pi \circ df_x : \tau(M)_x \longrightarrow \tau(N)_{f(x)}/\tau(Z)_{f(x)}$, where $\pi : \tau(N)_{f(x)} \to \tau(N)_{f(x)}/\tau(Z)_{f(x)}$ is the canonical projection. Indeed, $\overline{df_x}$ is an epimorphism if and only if for every $v \in \tau(N)_{f(x)}$ there is a $w \in \tau(M)_x$ such that $v - df_x(w) \in \tau(Z)_{f(x)}$. Thus f is always transverse to any open subset of N, because then the target of $\overline{df_x}$ is zero. Also if f is a submersion, then it is transverse to any submanifold Z of N. If Z is a point y in N, then the transversality condition (2) means that y is a regular value of f.

Exercise 1.2. Show that the set of points in $f^{-1}(Z)$ at which f is transverse to Z is an open subset of $f^{-1}(Z)$.

Definition 1.3. Two submanifolds Z_1 and Z_2 of a manifold N are called **transverse** (written $Z_1 \overline{\pitchfork} Z_2$) if the inclusion map $i : Z_1 \longrightarrow N$ is transverse to Z_2. Then the condition (2) means that, for every $x \in Z_1 \cap Z_2$,

$$\tau(Z_1)_x + \tau(Z_2)_x = \tau(N)_x,$$

since di_x is the inclusion of $\tau(Z_1)_x$ into $\tau(N)_x$.

In this case we also say that Z_1 and Z_2 are in a general position in N. It may be seen that a smooth map $f : M \longrightarrow N$ is transverse to a submanifold $Z \subset N$ if and only if the graph of f and $M \times Z$ are in general position in $M \times N$.

Note that we may interchange the role of Z_1 and Z_2 in this definition.

Example 1.4. Two curves in $\mathbb{R}^2$ which intersect at a point are non-transverse if the curves are tangent to each other at the point.

In all the results that we shall prove in this section, it is assumed that $f^{-1}(Z) \neq \emptyset$ whenever $f \overline{\pitchfork} Z$.

Lemma 1.5. *Let $f : M \longrightarrow N$ be a smooth map, and Z a submanifold of N with $\dim Z = m-k$ (or $\text{codim } Z = k$). Let $x \in f^{-1}(Z)$ and U be an open neighbourhood of $f(x)$ in N, and $g : U \longrightarrow \mathbb{R}^k$ a submersion such that $g^{-1}(0) = Z \cap U$ (U and g exist by Lemma 4.3, Part 1). Then $f \pitchfork_x Z$ if and only if x is a regular point* (or 0 is a regular value) *of $g \circ f$.*

Proof. We have $\ker dg_{f(x)} = \tau(Z)_{f(x)}$, since g is constant on $Z \cap U$. Then $f \pitchfork_x Z$ if and only if

$$df_x(\tau(M)_x) + \ker dg_{f(x)} = \tau(N)_{f(x)},$$

or equivalently, $d(g \circ f)_x = dg_{f(x)} \circ df_x$ is an epimorphism. To see this, take any $a \in \mathbb{R}^k$, then, since $dg_{f(x)}$ is an epimorphism, $a = dg_{f(x)}(u)$ for some $u \in \tau(N)_{f(x)}$, and $u = df_x(v)+w$ for some $v \in \tau(M)_x$ and $w \in \ker dg_{f(x)}$, therefore $a = dg_{f(x)} \circ df_x(v)$, and $d(g \circ f)_x$ is an epimorphism.

Conversely, if $d(g \circ f)_x$ is an epimorphism, then, for any $v \in \tau(N)_{f(x)}$, there is a $w \in \tau(M)_x$ such that $dg_{f(x)} \circ df_x(w) = dg_{f(x)}(v)$, so $v - df_x(w) \in \ker dg_{f(x)}$,and the above equality of vector spaces holds. □

Theorem 1.6. *If a smooth map $f : M \longrightarrow N$ is transverse to a submanifold Z of N, where none of M, N, and Z has boundary, then $f^{-1}(Z)$ is a submanifold of M, whose codimension in M is equal to the codimension of Z in N. In particular, if $\dim M = \text{codim } Z$, then $f^{-1}(Z)$ consists of isolated points only.*

Proof. It is sufficient to show that $f^{-1}(Z)$ is locally a submanifold, that is, each point x of $f^{-1}(Z)$ has an open neighbourhood V in M such that $V \cap f^{-1}(Z)$ is a manifold.

By Lemma 1.5, each $x \in f^{-1}(Z)$ is a regular point of $g \circ f$, and the open neighbourhood $V = f^{-1}(U)$ of x is such that $V \cap f^{-1}(Z) = (g \circ f)^{-1}(0)$. Therefore, by Theorem 4.8, Part 1, $V \cap f^{-1}(Z)$ is a submanifold of M of dimension $n - k$, where $n = \dim M$ and $k = \text{codim } Z$. □

Corollary 1.7. *The intersection of two transverse submanifolds Z_1 and Z_2 of N, where none of them has boundary, is a submanifold, and*

$$\text{codim } (Z_1 \cap Z_2) = \text{codim } Z_1 + \text{codim } Z_2.$$

Proof. This is a special case of the above theorem. □

Note that the transversality of two submanifolds depends on the dimension of the manifold where they are embedded. For example, the two coordinate axes are transverse in $\mathbb{R}^2$, but they are not in $\mathbb{R}^3$.

Lemma 1.8. *Let M be a manifold with boundary, and $\pi : M \longrightarrow \mathbb{R}$ is a smooth map with a regular value at 0, then $\pi^{-1}[0, \infty)$ is a manifold of M with boundary $\pi^{-1}(0)$*

Proof. The set $\pi^{-1}(0, \infty)$ is open in M, so it is a submanifold of M. A point $x \in M$ for which $\pi(x) = 0$ is a regular point, and so it has an open neighbourhood in M where π looks like the canonical submersion. The proof now follows, because the lemma is obviously true for the canonical submersion $\mathbb{R}^n \longrightarrow \mathbb{R}$. □

NOTATION. If M is a manifold with boundary and $f : M \longrightarrow N$ is a smooth map, then ∂f will denote the restriction map $f|\partial M : \partial M \longrightarrow N$.

Theorem 1.9. *Let M be a manifold with boundary, N a boundaryless manifold, Z a boundaryless submanifold of N, and $f : M \longrightarrow N$ a smooth map. Then, if both f and $\partial f = f|\partial M$ are transverse to Z, the inverse image $f^{-1}(Z)$ is a neat submanifold of M with boundary*

$$\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial M,$$

and the codimension of $f^{-1}(Z)$ in M equals that of Z in N.

Proof. The map $f|\mathrm{Int}\ M : \mathrm{Int}\ M \longrightarrow N$ is transverse to Z, and therefore $(f|\mathrm{Int}M)^{-1}(Z) = f^{-1}(Z) \cap \mathrm{Int}M$ is a boundaryless submanifold, by Theorem 1.6. Therefore it is necessary only to examine $f^{-1}(Z)$ in a neighbourhood of a point $x \in f^{-1}(Z) \cap \partial M$ (note that $f^{-1}(Z) = (f^{-1}(Z) \cap \mathrm{Int}M) \cup (f^{-1}(Z \cap \partial M))$. Applying Lemma 4.3, Part 1 to $f(x) \in Z$, we get an open neighbourhood P of $f(x)$ in N and a submersion $g : P \longrightarrow \mathbb{R}^k$ ($k = \mathrm{codim}\ \ Z$) such that $g^{-1}(0) = Z \cap P$. Then $(g \circ f)^{-1}(0) = f^{-1}(Z) \cap Q$, where $Q = f^{-1}(P)$. Let U be a coordinate neighbourhood of x and $\phi : U \longrightarrow \mathbb{R}^n_+$ be a coordinate system with $\phi(U \cap Q) = V$ an open set in $\mathbb{R}^n_+$ ($n = \dim\ M$). Let h denote the smooth map $g \circ f \circ \phi^{-1} : V \longrightarrow \mathbb{R}^k$. Then $f^{-1}(Z)$ will be a manifold with boundary near x if and only if $\phi(f^{-1}(Z) \cap Q \cap U) = h^{-1}(0)$ is a manifold with boundary near $\phi(x) = a$.

We have by Lemma 1.5,

$$f \overline{\pitchfork}_x Z \Leftrightarrow g \circ f \text{ is regular at } x \Leftrightarrow h \text{ is regular at } a.$$

Extend $h : V \longrightarrow \mathbb{R}^k$ to a smooth map $\widetilde{h} : \widetilde{V} \longrightarrow \mathbb{R}^k$ on an open set $\widetilde{V}$ of $\mathbb{R}^n$. Since $d\widetilde{h}_a = dh_a$, h is regular at a implies $\widetilde{h}$ is regular at a. This means that the intersection of $\widetilde{h}^{-1}(0)$ with some open neighbourhood of the regular point a is a boundaryless submanifold A of $\mathbb{R}^n$, since $\widetilde{h}$ is a smooth map of the boundaryless manifold $\widetilde{V}$. Without loss of generality, we may suppose that $A = \widetilde{h}^{-1}(0)$.

As $h^{-1}(0) = A \cap \mathbb{R}^n_+$, we must show that $A \cap \mathbb{R}^n_+$ is a manifold with boundary. For this purpose, let $\pi : A \longrightarrow \mathbb{R}$ be the restriction to A of the first coordinate function on $\mathbb{R}^n$. Then $A \cap \mathbb{R}^n_+ = \pi^{-1}[0, \infty)$. By Lemma 1.8, this will be a manifold with boundary if 0 is a regular value of π. Suppose that 0 is not a regular value of π. Then $\pi(z) = 0$ and $d\pi_z = 0$ for some point $z \in A$. Now $\pi(z) = 0$ implies $z \in A \cap \partial\mathbb{R}^n_+$, and, since $d\pi_z = \pi$ (π being linear), $d\pi_z = 0$ implies that the first coordinate of every vector in the tangent space $\tau(A)_z$ is zero, or $\tau(A)_z \subset \tau(\partial R^n_+)_z = \mathbb{R}^{n-1}$. Since $\widetilde{h}^{-1}(0) = A$, $\tau(A)_z$ is the kernel of $d\widetilde{h}_z$, and, since $dh_z = d\widetilde{h}_z$, $\ker dh_z = \tau(A)_z \subset \mathbb{R}^{n-1}$. This means that the maps $dh_z : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ and $d(\partial h)_z : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^k$ have the same kernel, because $d(\partial h)_z = dh_z|\mathbb{R}^{n-1}$. By transversality conditions both the linear maps are epimorphisms. But the dimension relation for linear maps says that $\dim\ker dh_z = n - k$, whereas $\dim\ker\ \ d(\partial h)_z = n - 1 - k$. This is a contradiction. Therefore 0 must be a regular value of h. □

Exercise 1.10. Let M be a manifold with boundary, N a manifold without boundary, Z a submanifold of N with boundary, and $f : M \longrightarrow N$ a smooth map. Then, if both f and $\partial f = f|\partial M$ are transverse to Z and f is transverse to ∂Z, the inverse image $f^{-1}(Z)$ is a submanifold of M with boundary

$$\partial(f^{-1}(Z)) = [f^{-1}(Z) \cap \partial M] \cup f^{-1}(\partial Z),$$

and the codimension of $f^{-1}(Z)$ in M equals that of Z in N.

Theorem 1.11 (**Generalization of Sard's Theorem**). *Let $f : M \to N$ be a smooth map, where $\partial M \neq \emptyset$ and $\partial N = \emptyset$. Then almost every point of N is a regular value of both $f : M \to N$ and $\partial f : \partial M \to N$.*

Proof. If $x \in \partial M$, then the restriction of the derivative map df_x to the subspace $\tau(\partial M)_x \subset (\tau(M))_x$ is the derivative map $d(\partial f)_x : \tau(\partial M)_x \to \tau(N)_x$. Therefore f is regular at x, if ∂f is so. Therefore a point $y \in N$ fails to be a regular value of both $f : M \to N$ and $\partial f : \partial M \to N$ only when it is a critical value of $f : \operatorname{Int}(M) \to N$ or $\partial f : \partial M \to N$. Now, since $\operatorname{Int}(M)$ and ∂M are both boundaryless manifolds, both the set of critical values have measure zero. Thus the complement of the set of common regular values for f and ∂f has measure zero, since it is the union of two sets of measure zero. $\square$

Exercise 1.12 (**Transitivity of transverse maps**). Let $f : M \longrightarrow N$ and $g : N \longrightarrow R$ be smooth maps, and A is a submanifold of R such that $g \overline{\pitchfork} A$. Then show that $f \overline{\pitchfork}\, g^{-1}(A)$ if and only if $(g \circ f) \overline{\pitchfork} A$.

In particular, if $M = N$ and f is a diffeomorphism, then $(g \circ f) \overline{\pitchfork} A$.

Theorem 1.13 (**Transversality Theorem**). *Suppose M, N, A, and B are manifolds, where only M has boundary, and A is a submanifold of N. Suppose $F : M \times B \longrightarrow N$ is a smooth map such that both F and ∂F are transverse to A. Suppose, for each $b \in B$, $f_b : M \longrightarrow N$ is the map $f_b(x) = F(x, b)$. Then. for almost all $b \in B$, both f_b and ∂f_b are transverse to A.*

Proof. By Theorem 1.9, $C = F^{-1}(A)$ is a manifold with boundary $\partial C = C \cap \partial(M \times B)$. Let $\pi : M \times B \longrightarrow B$ be the projection onto the second factor. We shall show that (1) if $b \in B$ is a regular value of $\pi|C$, then f_b is transverse to A, and (2) if $b \in B$ is a regular value of $\partial\pi|\partial C$, then ∂f_b is transverse to A. This will complete the proof, because, by Sard's theorem, almost every point b of B is a regular value of both the maps $\pi|C$ and $\partial\pi|\partial C$.

It is sufficient to prove only (1), because (2) follows from the fact that (1) is true for the special case of the boundaryless manifold ∂M.

Let x be any point of $f_b^{-1}(A)$. Then $f_b(x) = F(x, b) = a \in A$, and the transversality condition of F gives that

$$dF_{(x,b)}(\tau(M \times B)_{(x,b)}) + \tau(A)_a = \tau(N)_a.$$

Therefore, there is always a $u \in \tau(M \times B)_{(x,b)}$ for any given $v \in \tau(N)_a$ such that $dF_{(x,b)}(u) - v \in \tau(A)_a$. The problem here is to find a $w \in \tau(M)_x$ such that $df_b(w) - v \in \tau(A)_a$. Since $\tau(M \times B)_{(x,b)} = \tau(M)_x \times \tau(B)_b$, we may write $u = (r, t)$ where $r \in \tau(M)_x$ and $t \in \tau(B)_b$. If b is a regular value of $\pi|C : C \longrightarrow B$, then $d(\pi|C)_{(x,b)} : \tau(C)_{(x,b)} \longrightarrow \tau(B)_b$ is onto, and it is the restriction of $d\pi_{(x,b)} : \tau(M)_x \times \tau(B)_b \longrightarrow \tau(B)_b$ which is just the projection onto the second factor. Therefore for $t \in \tau(B)_b$ we can find a $(s, t) \in \tau(C)_{(x,b)}$, $s \in \tau(M)_x$, which is mapped onto t by $d\pi_{(x,b)}$. Since F maps C onto A, and $F(x, b) = a$, we have $dF_{(x,b)}(s, t) \in \tau(A)_a$. Then $w = r - s \in \tau(M)_x$ is a solution of our problem. To see this, first note that, since $F|(M \times \{b\}) = f_b$, we have $dF_{(x,b)}(w, 0) = df_b(w)$. Therefore

$$df_x(w) - v = dF_{(x,b)}(w, 0) - v = dF_{(x,b)}[(r, t) - (s, t)] - v$$

$$= [dF_{(x,b)}(r,t) - v] - dF_{(x,b)}(s,t) = [dF_{(x,b)}(u) - v] - dF_{(x,b)}(s,t) \in \tau(A)_a.$$

This completes the proof. □

A consequence of the transversality theorem is that transverse maps are generic, in the sense that almost all C^∞ maps $M \longrightarrow N$ are transverse to any submanifold A of N. We show this first for the case when $N = \mathbb{R}^m$.

Corollary 1.14. *Any smooth map $f : M \longrightarrow \mathbb{R}^m$ is arbitrarily close to a smooth map $g : M \longrightarrow \mathbb{R}^m$ which is transverse to any boundaryless submanifold A of $\mathbb{R}^m$. Moreover, f is homotopic to g by a small homotopy.*

Proof. Let B be an open ball about the origin in $\mathbb{R}^m$. Define $F : M \times B \longrightarrow \mathbb{R}^m$ by $F(x, b) = f(x) + b$. For a fixed $x \in M$, F simply translates the ball B, and so it a submersion on $\{x\} \times B$. Then, both F and ∂F are submersions, because the Jacobian matrix of each contains the matrix of this translation as a submatrix. Therefore F and ∂F are transverse to any boundaryless submanifold A of $\mathbb{R}^m$. Then Theorem 1.13 implies that for almost every $b \in B$, the map $f_b(x) = f(x) + b$ is transverse to A. Choosing such a b, f may be deformed into the transverse map $g = f_b$ by the homotopy $H : M \times I \longrightarrow \mathbb{R}^m$ given by $H(x, t) = f(x) + tb$. Finally, the map g may be made arbitrarily closed to f by making the radius of B sufficiently small. □

The next theorem deals with the general case.

Theorem 1.15 (Transversality Homotopy Theorem). *For any smooth map $f : M \longrightarrow N$, $\partial N = \emptyset$, and any boundaryless submanifold A of N, there is a smooth map $g : M \longrightarrow N$ such that both g and ∂g are transverse to A, and f is homotopic to g.*

Proof. Embed N in some $\mathbb{R}^m$, and let $N(\epsilon) = \{x \in \mathbb{R}^m | d(x, N) < \epsilon(x)\}$ be an ϵ-neighbourhood of N in $\mathbb{R}^m$ with retraction $r : N(\epsilon) \longrightarrow N$, where ϵ is a smooth positive function on N. Let B be the open unit ball in $\mathbb{R}^m$. Define $F : M \times B \longrightarrow N$ by $F(x, b) = r[f(x) + \epsilon(f(x))\dot{b}]$. Since r is a retraction onto M, $F(x, 0) = f(x)$. Also, both F and ∂F are submersions on $\{x\} \times B$ for a fixed x, being the composition of two submersions $b \mapsto f(x) + \epsilon(f(x))b$ and r. Since each point of $M \times B$, and each point of $\partial M \times B$, lies on a submanifold $\{x\} \times B$, both F and ∂F are submersions.

Therefore both F and ∂F are transverse to any boundaryless submanifold A of N, and hence both f_b and ∂f_b are transverse to A for all most all $b \in B$. Finally, f is homotopic to each such f_b by homotopy $H : M \times I \longrightarrow N$ given by $H(x, t) = F(x, tb)$. □

Theorem 1.16 (Extension Theorem). *Let $f : M \longrightarrow N$ be a smooth map, K a closed subset of M, and A a closed subset and a submanifold of N. Let both A and N be without boundary. Let $f \pitchfork A$ on K and $\partial f \pitchfork A$ on $K \cap \partial M$. Then there is a smooth map $g : M \longrightarrow N$ such that f is smoothly homotopic to g, $g \pitchfork A$, $\partial g \pitchfork A$, and $f = g$ on a neighbourhood of K.*

Proof. First show that $f \pitchfork A$ on a neighbourhood of K by considering the following two cases.

(1) $x \in K$ and $x \notin f^{-1}(A)$, and

(2) $x \in K \cap f^{-1}(A)$.

In case (1), $M - f^{-1}(A)$ is an open neighbourhood of x (as A is closed) on which f is obviously transverse to A. In case (2), there is a neighbourhood U of $f(x)$ in N and a submersion $\phi : U \longrightarrow \mathbb{R}^k$ such that $f \overline{\pitchfork} A$ implies that $\phi \circ f$ is regular at x, and hence in a neighbourhood of x. Thus $f \overline{\pitchfork} A$ on a neighbourhood of every point of K, so $f \overline{\pitchfork} A$ on a neighbourhood U of K.

Next, construct a map $\alpha : M \longrightarrow [0, 1]$ such that $\alpha = 1$ on $M - U$, and $\alpha = 0$ on a neighbourhood of K. The construction of α may be seen easily by applying the Smooth Urysohn's Lemma (Lemma 1.17, Part 2) to the open neighbourhood $M - K$ of the closed set $M - U$. Define $\beta = \alpha^2$. Then $d\beta_x = 2\alpha(x) \cdot d\alpha_x$. Therefore $d\beta_x = 0$ whenever $\alpha(x) = 0$, that is, $\beta(x) = 0$. Now, the proof of Transversality Homotopy Theorem (Theorem 1.15) gives a smooth map $F : M \times B \longrightarrow N$, B an open ball in some $\mathbb{R}^m$, such that $F(x, 0) = f(x)$, both F and ∂F are submersions, and, for fixed $x \in M$, the map $b \mapsto F(x, b)$ is a submersion $B \longrightarrow N$. Define a map $G : M \times B \longrightarrow N$ by $G(x, b) = F(x, \beta(x) \cdot b)$. Then $G \overline{\pitchfork} A$.

To see this, suppose that $(x, b) \in G^{-1}(A)$ and $\beta(x) \neq 0$. Then the map $B \longrightarrow N$ given by $b \mapsto G(x, b)$ is a submersion, because it is the composition of a diffeomorphism $b \mapsto \beta(x) \cdot b$ and a submersion $b \mapsto F(x, b)$. Therefore G is a submersion at (x, b), and hence $G \overline{\pitchfork} A$ at (x, b) in this case. Next, if $\beta(x) = 0$ the conclusion can be arrived at by computing $dG_{(x,b)}$ at a point $(v, w) \in \tau(M)_x \times \tau(B)_b = \tau(M)_x \times \mathbb{R}^m$. Note that $G = F \circ H$, where $H : M \times B \longrightarrow M \times B$ is given by $H(x, b) = (x, \beta(x) \cdot b)$. Then

$$dH_{(x,b)}(v, w) = (v, \beta(x) \cdot w + d\beta_x(v) \cdot b) = (v, 0),$$

since $\beta(x) = 0$ and $d\beta_x = 0$, and we have

$$dG_{(x,b)}(v, w) = dF_{(x,0)}(v, 0) = df_x(v),$$

as $F \mid M \times 0 = f$. This implies that $\operatorname{Im} dG_{(x,b)} = \operatorname{Im} df_x$. But, if $\beta(x) = 0$, then $\alpha(x) = 0$, so x belongs to the neighbourhood U of K, and therefore $f \overline{\pitchfork} A$ at x. This means that $G \overline{\pitchfork} A$, since $\operatorname{Im} dG_{(x,b)} = \operatorname{Im} df_x$, and $G(x, b) = f(x)$ for $x \in U$.

Similarly, it can be shown that $\partial G \overline{\pitchfork} A$. Therefore, by Transversality Theorem (Theorem 1.13), there is a $b \in B$ such that the map $g : M \longrightarrow N$ given by $g(x) = G(x, b)$, and the map ∂g are both transverse to A. Also, f is homotopic to g by homotopy $H : M \times I \longrightarrow N$ given by $H(x, t) = G(x, tb) = F(x, t\beta(x) \cdot b)$. Moreover, if x is in the neighbourhood U of K on which $\alpha = 0$, then $g(x) = G(x, b) = F(x, 0) = f(x)$. □

Corollary 1.17. *If for a smooth map $f : M \longrightarrow N$, the restriction to boundary $\partial f : \partial M \longrightarrow N$ is transverse to A, where $\partial N = \emptyset$ and $\partial A = \emptyset$, then there is a smooth map $g : M \longrightarrow N$ homotopic to f such that $\partial f = \partial g$ and $g \overline{\pitchfork} A$.*

Proof. This is a special case of the previous theorem, since the boundary ∂M is always closed in M. □

2. Orientation

Let V be a finite dimensional real vector space, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be two ordered bases of V. Then there is a unique linear map $L : V \to$

V such that $L(\alpha_i) = \beta_i$. The columns of the matrix of L are the components of the vectors $L(\alpha_1), \ldots, L(\alpha_n)$ with respect to the basis $\alpha_1, \ldots, \alpha_n$. Since $\beta_1, \ldots, \beta_n$ are linearly independent, the matrix of L is non-singular. We say that the bases α and β are equivalently oriented if det $L > 0$. This defines an equivalence relation and the set of all ordered bases is partitioned into two disjoint classes. We denote the equivalence class of $(\alpha_1, \ldots, \alpha_n)$ by $[\alpha_1, \ldots, \alpha_n]$, and it is called an **orientation of** V. Note that the ordering of the vectors is important here. We pick up a class arbitrarily and assign a positive sign to it, and a negative sign to the other class. The vector space V with an ordered basis α is called positively or negatively oriented depending on which orientation class α belongs. Thus each vector space with positive dimension has precisely two orientations. If the vector space is of dimension zero, we denote its orientation as one of the symbols $+1$ and -1.

Let $J : V \longrightarrow W$ be a linear isomorphism between oriented vector spaces. If α and β are bases of V, and $L : V \longrightarrow V$ is a linear transformation sending α to β, then $J \circ L \circ J^{-1}(J(\alpha)) = J(\beta)$. Since $\det L = \det J \circ L \circ J^{-1}$, the sign of the orientation of $J(\alpha)$ is either the same as the sign of the orientation of α, or opposite. Therefore the isomorphism J is either **orientation preserving** or **orientation reversing**. We may define sign $J =$ sign $\alpha \cdot$ sign $J(\alpha)$, where α is a basis of V, so that J is orientation preserving or reversing according as sign J is $+1$ or -1.

The standard orientation of an Euclidean space $\mathbb{R}^n$, $n \geq 1$, is given by any basis whose coordinate matrix has positive determinant. The standard orientation of $\mathbb{R}^0$ is the number $+1$.

Recall from Theorem 6.2, Part 1 that the tangent bundle $\tau(M)$ of a manifold M of dimension n is a manifold of dimension $2n$ with a smooth atlas $\{(\pi^{-1}(U_i), \tau_{\phi_i})\}$, where

$$\tau_{\phi_i} : \pi^{-1}(U_i) \to \phi_i(U_i) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n,$$

, corresponding to a smooth atlas $\{(U_i, \phi_i)\}$ of M.

Definition 2.1. An **orientation of the tangent bundle** $\tau(M)$ is a family

$$\omega = \{\omega_x \ : \ \omega_x \text{is an orientation of} \tau(M)_x, \quad x \in M\}$$

such that there exists a smooth atlas $\Phi = \{(U_i, \phi_i)\}$ of M with the condition that, for $x \in U_i$, the isomorphism $d\phi_x : (\tau(M)_x, \omega_x) \to (\mathbb{R}^{2n}, \lambda)$, where λ is the standard orientation of $\mathbb{R}^{2n}$, is orientation preserving.

A manifold M is called **orientable** if its tangent bundle $\tau(M)$ ia orientable. If ω is an orientation of $\tau(M)$, then we say that (M, ω) is an oriented manifold. In this case M admits another orientation $-\omega$ defined by $(-\omega)_p = -\omega_p, p \in M$.

Note that for any coordinate chart (U, ϕ) in M with $\phi = (x_1, \ldots, x_n)$, the linearly independent vector fields $\partial/\partial x_1, \ldots, \partial/\partial x_n$ on U define an orientation on U. The orientations defined in this way by two coordinate charts (U, ϕ) and (V, ψ) agree on $U \cap V$ if and only if $\det(\psi \circ \phi^{-1}) > 0$, and they define an orientation on $U \cup V$. This consideration leads us to the following alternative definition of orientation of a manifold.

Definition 2.2. A manifold M is **orientable** if there is an atlas $\Phi = \{(U_i, \phi_i)\}$ such that whenever $U_i \cap U_j \neq \emptyset$ the Jacobian matrix of $\psi_j \circ \phi_i^{-1}$ has strictly positive determinant at every point of $\phi_i(U_i \cap U_j)$. The atlas Φ is called an oriented atlas. A manifold is oriented if an oriented atlas has been chosen for it.

3. Boundary and preimage orientations

Here we describe two standard orientations, namely, boundary orientation and preimage orientation. They will be in force throughout the rest of the chapter.

Boundary orientation. Suppose that $\dim M = n \geq 1$. Then, since

$$\text{codim} \quad \partial M = 1,$$

at each point $x \in \partial M$, there are exactly two unit vectors in $\tau(M)_x$ which are orthogonal to $\tau(\partial M)_x$. One of them is inward pointing and the other is outward pointing. Here is their precise definition. At the origin $0 \in \mathbb{R}^n_+$, the unit vector $e_1 = (1, 0, \dots, 0)$ is the inward pointing normal vector to $\partial\mathbb{R}^n_+$, and $-e_1$ is the outward pointing normal vector. If $\phi : U \longrightarrow \mathbb{R}^n_+$ is a coordinate system with $\phi(x) = 0$, then $(d\phi_x)^{-1}(e_1)$ is the inward pointing normal vector to ∂M at x, and its negative is outward pointing. This distinction between inward and outward directions does not depend on the choice of ϕ. Because if ϕ' is another compatible coordinate system, then the isomorphism $d(\phi'\phi^{-1})_0$ maps the half space $\mathbb{R}^n_+$ onto itself (Lemma 7.1 Part 1). The boundary orientation on ∂M is defined as follows. If $\alpha = \{\alpha_1, \dots, \alpha_n\}$ is an ordered basis of $\tau(\partial M)_x$, then sign α is the sign of the ordered basis $\{\nu_x, \alpha\} = \{\nu_x, \alpha_1, \dots, \alpha_n\}$ of $\tau(M)_x$, where ν_x is the outward normal vector at x. It can be seen easily that this defines an orientation on ∂M. If $\dim M = 1$, then the orientation of the zero-dimensional vector space $\tau(\partial M)_x$ is the sign of the basis $\{\nu_x\}$ of $\tau(M)_x = \mathbb{R}$, $x \in \partial M$.

Remark 3.1. The orientation of $\tau(\partial M)_x$ is obtained from the direct sum decomposition $\tau(M)_x = \langle \nu_x \rangle \oplus \tau(\partial M)_x$. It is not necessary to take the outward unit normal vector ν_x in the definition of the boundary orientation. We might just as well replace ν_x by any outward pointing vector, which is a vector like $r\nu_x + w$, where $r > 0$, and $w \in \tau(\partial M)_x$, and get the same orientation of ∂M.

Example 3.2. The standard orientation of the closed unit disk D^2 induces counterclockwise orientation on the boundary circle S^1.

Example 3.3. The standard orientation of the unit interval $0 \leq t \leq 1$ induces the orientation -1 and $+1$ at the end points 0 and 1 respectively.

Example 3.4. Let M be an oriented manifold without boundary, and I be the unit interval $[0, 1]$ with the standard orientation. Then the product $M \times I$ has two boundary components $M_0 = M \times \{0\}$ and $M_1 = M \times \{1\}$, and each of them is diffeomorphic to M. At a point $(x, 0) \in M_0$, the outward normal vector $\nu_{(x,0)}$ is $(0, -1) \in \tau(M)_x \times \tau(I)_0$, and at a point $(x, 1) \in M_1$, the outward normal vector $\nu_{(x,1)}$ is $(0, 1) \in \tau(M)_x \times \tau(I)_1$. Therefore if α is an ordered basis of $\tau(M)_x$, the signs of the induced orientation of M_0 and M_1 are given respectively as

$$\text{sign } (\nu_{(x,0)}, \alpha) = \text{ sign } (-1) \cdot \text{ sign } \alpha = - \text{ sign } \alpha.$$

$$\text{sign } (\nu_{(x,1)}, \alpha) = \text{ sign } (1) \cdot \text{ sign } \alpha = \text{ sign } \alpha.$$

Thus the induced orientations on M_0 and M_1 are opposite, and we may write

$$\partial(M \times I) = M_1 \cup (-M_0).$$

Preimage orientation. Let M, N, and A be oriented manifolds, where A is a submanifold of N, and A and N are without boundary. Let $f : M \longrightarrow N$ be a smooth map with $f \overline{\pitchfork} A$ and $\partial f \overline{\pitchfork} A$.transverse to A. Then the manifold $B = f^{-1}(A)$ receives a natural orientation from the orientations on M, N, and A. This orientation on B, which is called the preimage orientation induced by f, is defined as follows.

Let $x \in M$ and $y = f(x) \in A$. Then, $\tau(B)_x = df_x^{-1}(\tau(A)_y)$ is a subspace of the vector space $\tau(M)_x$. Let $\nu(B)_x$ be the orthogonal complement of $\tau(B)_x$ in $\tau(M)_x$ so that

$$\tau(M)_x = \nu(B)_x \oplus \tau(B)_x. \tag{3.1}$$

Then, $df_x(\tau(M)_x) = df_x(\nu(B)_x) + df_x(\tau(B)_x)$. Substituting this in the transversality condition $\tau(N)_y = df_x(\tau(M)_x) + \tau(A)_y$, we get

$$\tau(N)_y = df_x(\nu(B)_x) \oplus \tau(A)_y. \tag{3.2}$$

The sum is direct, because the dimensions of both sides are equal (note that, since $\ker df_x \subset \tau(B)_x$, df_x maps $\nu(B)_x$ isomorphically onto its image). The orientations of $\tau(N)_y$ and $\tau(A)_y$ induce an orientation of $df_x(\nu(B)_x)$ via the direct sum decomposition (2), this induces an orientation of $\nu(B)_x$ via the isomorphism df_x, finally, the orientations of $\tau(M)_x$ and $\nu(B)_x$ induce an orientation of $\tau(B)_x$ via the decomposition (1). In this way, we may define orientation on each tangent space $\tau(B)_x$ smoothly, because df_x varies smoothly with x.

With this knowledge, we can summarize the rule for finding the preimage orientation as follows. If α, β, and γ are the orientations of M, A, and N respectively, then the preimage orientation ω of B is given by

$$\text{sign } \ \omega = \frac{\text{sign } \ \alpha \cdot \text{sign } \ \beta}{\text{sign } \ \gamma}. \tag{3.3}$$

Remark 3.5. Note that in this definition of the preimage orientation, the orthogonality of the complement of $\tau(B)_x$ is unnecessary. In fact, for any complement P of $\tau(B)_x$ in $\tau(M)_x$, the equations $\tau(M)_x = P \oplus \tau(B)_x$ and $df_x(P) \oplus \tau(A)_y = \tau(N)_y$ will define the same preimage orientation on B.

Recall that if M is a manifold with boundary ∂M, and if $f : M \longrightarrow N$ and $\partial f : \partial M \longrightarrow N$ are both transverse to a submanifold A of N, where both N and A are without boundary, then $B = f^{-1}(A)$ is a manifold with boundary

$$\partial B = f^{-1}(A) \cap \partial M.$$

Then ∂B receives two orientations, one as the preimage of A under $\partial f : \partial M \longrightarrow N$, and the other as the boundary of B. The follows lemma shows that these two orientations are the same if codim A is even.

Lemma 3.6. $\partial(f^{-1}(A)) = (-1)^{\text{codim } A}(\partial f)^{-1}(A)$.

Proof. At any point $x \in B$, $\tau(\partial B)_x$ is a subspace of $\tau(\partial M)_x$. Let P be a complement of $\tau(\partial B)_x$ in $\tau(\partial M)_x$ so that

$$\tau(\partial M)_x = P \oplus \tau(\partial B)_x \tag{3.4}$$

Therefore, since $\tau(\partial B)_x = \tau(B)_x \cap \tau(\partial M)_x$, $P \cap \tau(B)_x = 0$. This means that P is also a complement of $\tau(B)_x$ in $\tau(M)_x$, and we have the direct sum decomposition

$$\tau(M)_x = P \oplus \tau(B)_x, \tag{3.5}$$

since $\dim P + \dim \tau(B)_x = \dim \tau(M)_x$. Thus P is complementary to both $\tau(B)_x$ and $\tau(\partial B)_x$. Now, df_x and $d(\partial f)_x$ agree on P, because $P \subset \tau(\partial M)_x$. Therefore P receives the same orientation by the maps df_x and $d(\partial f)_x$, via the decomposition

$$\tau(N)_{f(x)} = df_x(P) \oplus \tau(A)_{f(x)}.$$

Then decomposition (3.5) (resp. (3.4)) defines the preimage orientation of B (resp. ∂B) induced by f (resp. ∂f) (see Remark 3.5).

The boundary orientation of ∂B induced from the orientation of B is defined by the decomposition $\tau(B)_x = \langle \nu_x \rangle \oplus \tau(\partial B)_x$, where ν_x is the outward normal to ∂B in B, and $\langle \nu_x \rangle$ denotes the one-dimensional space spanned by it, oriented so that $\{v_x\}$ is a positively oriented basis. This v_x may not be orthogonal to $\tau(\partial M)_x$. But we may suppose that v_x is an outward pointing vector (see Remark 3.1) so that the orientations of $\tau(\partial M)_x$ and $\tau(M)_x$ are related by the direct sum relation $\tau(M)_x = \langle v_x \rangle \oplus \tau(\partial M)_x$. Substituting the preimage orientations of B and ∂B into this, we obtain

$$P \oplus \tau(B)_x = \langle \nu_x \rangle \oplus P \oplus \tau(\partial B)_x.$$

Then, further imposition of the boundary orientation of ∂B yields

$$P \oplus \langle \nu_x \rangle \oplus \tau(\partial B)_x = \langle \nu_x \rangle \oplus P \oplus \tau(\partial B)_x = (-1)^k P \oplus \langle \nu_x \rangle \oplus \tau(\partial B)_x,$$

where the last equality is obtained by making $k = \dim P$ number of transpositions to move ν_x from left to right. Therefore, we have $\tau(\partial B)_x = (-1)^k \tau(\partial B)_x$, where on the left hand side $\tau(\partial B)_x$ has the boundary orientation and on the right hand side it has the preimage orientation. Since $\dim P = \operatorname{codim} \ B = \operatorname{codim} \ A$, this completes the proof. □

4. Intersection numbers, and Degrees of maps

Let M, N, and A be oriented manifolds without boundary, where M is compact, and A is a closed submanifold of N, such that

$$\dim M + \dim A = \dim N. \tag{4.1}$$

Now, if $f : M \longrightarrow N$ is a smooth map transverse to A, then $f^{-1}(A)$ is a finite set of points, since it is compact and its dimension is zero (see that $\operatorname{codim} \ f^{-1}(A) = \operatorname{codim} \ A = \dim M$ by the dimension condition (4.1)). If $x \in f^{-1}(A)$, then the transversality condition at x and the dimension condition (4.1) imply that we have a direct sum decomposition

$$df_x(\tau(M)_x \oplus \tau(A)_{f(x)} = \tau(N)_{f(x)}, \tag{4.2}$$

and df_x is an isomorphism onto its image.

Definition 4.1. The preimage orientation at a point $x \in f^{-1}(A)$ is called the **orientation number** of f, and is denoted by $n(f, x)$. This number is $+1$ if in the direct sum decomposition (4.2), the orientation on $df_x(\tau(M)_x)$ plus the orientation on $\tau(A)_{f(x)}$ (in this order) is the prescribed orientation on $\tau(N)_{f(x)}$, and it is -1 otherwise.

An alternative definition of $n(f, x)$ may be obtained in the following way. The composition $\pi \circ df_x$

$$\tau(M)_x \xrightarrow{df_x} \tau(N)_{f(x)} \xrightarrow{\pi} \tau(N)_{f(x)}/\tau(A)_{f(x)},$$

where π is the canonical projection, is an isomorphism. Then $n(f, x)$ is $+1$ or -1 according to whether this isomorphism $\pi \circ df_x$ preserves or reverses orientation.

The **intersection number** $I(f, A)$ is defined to be the sum of the orientation numbers $n(f, x)$ over all $x \in f^{-1}(A)$.

If $f : M \longrightarrow N$ is any smooth map, then Theorem 1.15 says that there is a smooth map $g : M \longrightarrow N$ homotopic to f such that both g and ∂g are transverse to A. In this case the intersection number $I(f, A)$ is defined to be the intersection number $I(g, A)$. That this definition in independent of the choice of g will be seen in Lemma 4.4 below.

Exercise 4.2. Let $f : M \longrightarrow N$ and $g : M \longrightarrow (N - A)$ be smooth maps homotopic in N. Then show that $I(f, A) = 0$.

Theorem 4.3 (**Extendability Theorem**)**.** *Let M, N, and A be oriented manifolds, where M is compact with boundary, and both N and A are boundaryless. Let A be a closed subset and a submanifold of N such that*

$$\dim \partial M + \dim A = \dim N.$$

Let $f : \partial M \longrightarrow N$ be a smooth map which extends to a smooth map $g : M \longrightarrow N$ such that both g and ∂g are transverse to A. Then $I(f, A) = 0$.

Note that f may be a map from a component of ∂M into N.

Proof. In view of the given dimension condition, $g^{-1}(A)$ is a compact oriented one-dimensional manifold with boundary $\partial(g^{-1}(A)) = g^{-1}(A) \cap \partial M = f^{-1}(A)$, which consists of pairs of points with orientation numbers $+1$ and -1. Consequently, $I(f, A) = 0$. □

Lemma 4.4. *Let M, N, and A be oriented boundaryless manifolds, where M is compact. Let A be a closed subset and a submanifold of N. Let g_0 and g_1 be smooth maps from M into N which are homotopic and both transverse to A. Then $I(g_0, A) = I(g_1, A)$.*

Proof. If $G : M \times I \longrightarrow N$ is a homotopy between g_0 and g_1, then $I(\partial G, A) = 0$ by the above lemma. Now, by Example 3.4, $\partial(M \times I) = M_1 - M_0$, where M_0 and M_1 are diffeomorphic copies of M, and ∂G agrees with g_0 and g_1 on M_0 and M_1 respectively. Therefore $(\partial G)^{-1}(A) = g_1^{-1}(A) - g_0^{-1}(A)$, and hence

$$I(g_1, A) - I(g_0, A) = I(\partial G, A) = 0.$$

□

In the special case when $\dim M = \dim N$, A is a point $y \in N$, and N is connected, the intersection number $I(f, \{y\})$ is called the **degree of** f, and denoted by $\deg f$.

Here y is a regular value of f, and for $x \in f^{-1}(y)$, $n(f, x)$ is $+1$ or -1 according as df_x preserves or reverses orientation. Then

$$\deg\ f = \sum_{x \in f^{-1}(y)} n(f, x).$$

In general, we may allow y to be any point of N (not necessarily a regular value of f), and define deg f by the above formula. This is justified by the following lemma.

Lemma 4.5. *The number* $\deg f$ *is the same for all* $y \in N$.

Proof. Given y, we may find a smooth map $g : M \longrightarrow N$ such that y is a regular value of g, and g is homotopic to f. Then $g^{-1}(y)$ is a finite set $\{x_1, \ldots, x_k\}$, say. Because $\dim M = \dim N$, we may invoke the local submersion theorem to find disjoint open neighbourhoods U_i of x_i such that g maps each U_i diffeomorphically onto an open neighbourhood V of y, and $g^{-1}(V) = U_1 \cup \cdots \cup U_k$ (disjoint union). Then, for any $z \in V$, $g^{-1}(z)$ consists of k points, and, for all x' in an U_i, the orientation number $n(g, x')$ is the same. This means that the map $N \longrightarrow \mathbb{Z}$ given by $y' \mapsto I(g, \{y'\})$ is locally constant. Since N is connected, this map must be constant on the whole of N. $\square$

For example, The identity map of M has degree $+1$, And the anti-podal map $S^n \longrightarrow S^n$, sending x to $-x$, has degree $(-1)^{n+1}$.

Exercise 4.6. Let M, N, and P be connected boundaryless manifolds of the same dimension. Then for smooth maps $f : M \longrightarrow N$ and $g : N \longrightarrow P$ show that $\deg(g \circ f) = \deg f \cdot \deg g$.

Lemma 4.7. *Represent the circle S^1 as the set of complex numbers z with $|z| = 1$, and let m be an integer. Then the map f of the circle S^1 onto itself, given by $z \mapsto z^m$, has degree m.*

Proof. If $m > 0$ (resp. $m < 0$), the image point $f(z)$ moves around the circle in the counterclockwise (resp. clockwise) sense m times as z moves around the circle once in the counterclockwise sense. Therefore the inverse image $f^{-1}(z)$ of a point $z \in S^1$ contains $|m|$ number of points, unless $m = 0$. In terms of the coordinate systems arising from the exponential map $\mathbb{R} \longrightarrow S^1$ given by $\theta \mapsto \exp(i\theta)$. local representation of f is $\theta \mapsto m\theta$. Therefore f is regular everywhere if $m \neq 0$. If $m > 0$, f is orientation preserving, and $\deg f$ is the number of points in $f^{-1}(z)$ which is m. If $m < 0$, f is orientation reversing, and $\deg f$ is $-|m| = m$. If $m = 0$, then f is a constant map, and so $\deg f = 0$. $\square$

Thus, given any integer m, there is a smooth map $S^1 \longrightarrow S^1$ whose degree is m.

Example 4.8. Any smooth map $f : S^1 \longrightarrow S^1$ may be written as

$$f(\exp(i\theta)) = \exp(ig(\theta)),$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth map. Then, since θ and $\theta + 2\pi$ represent the same point of the circle, we must have $g(\theta + 2\pi) = g(\theta) + 2k\pi$ for all θ. where k is a fixed integer which is positive or negative according to whether f is orientation preserving or reversing. In particular, $g(2\pi) = g(0) + 2k\pi$. Thus as z moves round the circle once, the image $f(z)$ moves round the circle k times. Thus $\deg f = k$.

Theorem 4.9. *Two maps $f_0, f_1 : S^1 \longrightarrow S^1$ are homotopic if and only if they have the same degree.*

Proof. We already know that the condition is necessay. To prove the sufficiency, suppose that $\deg f_0 = \deg f_1$. Then, as in Example 4.8, we may write, for $j = 1, 2$, f_j as $f_j(\exp(i\theta)) = \exp(ig_j(\theta))$, where $g_j : \mathbb{R} \longrightarrow \mathbb{R}$ are smooth maps such that $g_j(\theta + 2\pi) = g_j(\theta) + 2k\pi$. Then g_0 and g_1 are homotopic by homotopy

$$g_t = (1-t)g_0 + tg_1, \quad 0 \le t \le 1.$$

Then $g_t(\theta + 2\pi) = g_t(\theta) + 2k\pi$ and so $\exp(ig_t(\theta))$ is a homotopy between f_0 and f_1. □

This theorem is a special case of the Hopf degree theorem. One-half of the Hopf degree theorem is contained in the extendability theorem, which in terms of the degree reads as follows.

Theorem 4.10. *Let M be a compact oriented manifold which is the boundary of a manifold P, and N another oriented boundaryless manifold of the same dimension as M. If a smooth map $f : M \longrightarrow N$ extends to a smooth map on all of P, then* $\deg f = 0$.

The Hopf degree theorem says that $f : M \longrightarrow S^n$ extends to a smooth map on all of P if and only if $\deg f = 0$. We omit the proof, and look at a simple application of Theorem 4.10 instead.

Let $p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$ be a monic polynomial with complex coefficients. Consider a family of polynomials of $p_t(z)$, $0 \le t \le 1$, given by

$$p_t(z) = tp(z) + (1-t)z^m = z^m + t(a_1 z^{m-1} + \cdots + a_m).$$

Since

$$\lim_{z \to \infty} \frac{p_t(z)}{z^m} = 1,$$

there is a closed ball B of sufficiently large radius such that none of the polynomials $p_t(z)$ vanish on ∂B. Therefore the homotopy $p_t/|p_t| : \partial B \longrightarrow S^1$ is defined for all t, and so $\deg(p/|p|) = \deg(p_0/|p_0|) = m$, by Lemma 4.7. This implies that p has at least one zero inside B. Otherwise, the map $p/|p| : \partial B \longrightarrow S^1$ will extend on all of B, and $\deg(p/|p|)$ will be zero, by the extendability property. This proves

Theorem 4.11 (Fundamental Theorem of Algebra). *Any non-constant complex polynomial has a zero.*

Let M, N, and A be oriented boundaryless manifolds, where M is a compact submanifold of N. Let A be a closed subset and a submanifold of N of complementary dimension (so that (4.1) is satisfied). Let $i : M \longrightarrow N$ be the inclusion map, and $M \pitchfork A$. Then the **intersection number** $I(M, A; N)$ is defined to be the number $I(i, A)$. This is the sum of the orientation numbers $+1$ and -1 of the points $x \in M \cap A$, where the number for x is $+1$ if the orientations of M and A (in this order) at x give the orientation of N at x, otherwise the number is -1.

We often write $I(M, A; N)$ simply as $I(M, A)$, when it is not necessary to mention the ambient manifold N.

Exercise 4.12. Show that if M and A are compact submanifolds of N of dimensions n and k respectively such that $\dim N = n + k$, then

$$I(M, A) = (-1)^{nk} I(A, M).$$

In particular, if $\dim N = 2 \dim M$, then the self-intersection number $I(M, M)$ is defined. Moreover, if $\dim M$ is odd, then $I(M, M) = 0$.

Definition 4.13. Let M be a compact connected manifold without boundary, $\pi : \tau(M) \longrightarrow M$ be the tangent bundle of M, and $i : M \longrightarrow \tau(M)$ be the zero section. Identify M with the zero section. Then the **Euler characteristic** of M, denoted by $\chi(M)$, is defined to be the number $I(M, M) = I(M, M; \tau(M))$.

Definition 4.14. Let $g : M \longrightarrow \tau(M)$ be a smooth vector field transverse to the zero-section $i(M)$. Let $x \in M$ be a zero of g (i.e. $x \in g^{-1}(i(M))$). Then the orientation number $n(g, x)$ is called the **index of the vector field** g **at** x, and is denoted by Ind ${}_x g$.

Lemma 4.15. *A compact oriented manifold M without boundary admits a vector field $g : M \longrightarrow \tau(M)$ transverse to the zero-section $i(M)$ such that*

$$\chi(M) = \sum_{x \in g^{-1}(i(M))} \text{Ind}\ {}_x g.$$

Proof. Approximate the zero section $i : M \longrightarrow \tau(M)$ by a smooth map

$$f : M \longrightarrow \tau(M)$$

homotopic to i and transverse to the zero section $i(M)$ (Theorem 1.15). If the approximation is sufficiently small, then the map $\pi \circ f : M \longrightarrow M$ is a diffeomorphism homotopic to Id (Theorem 6.2(5) Part 2), and the map $g = f \circ (\pi \circ f)^{-1} : M \longrightarrow \tau(M)$ is a smooth section transverse to the zero section $i(M)$ (Exercise 1.12 in page 4), and homotopic to i. Therefore

$$\chi(M) \;=\; I(i, M) \;=\; I(g, M) \;=\; \sum_{x \in g^{-1}(i(M))} \text{Ind}\ {}_x g.$$

□

Theorem 4.16. *If M admits a nowhere vanishing vector field, then its Euler characteristic $\chi(M) = 0$.*

Proof. If $g : M \longrightarrow \tau(M)$ is a nowhere vanishing vector field, then g is transverse to the zero-section $i(M)$. Also g is homotopic to the zero section i by the homotopy given by $h_t(x) \mapsto tf(x)$. Therefore

$$\chi(M) = I(i, M) = I(g, M) = 0.$$

□

We may compute $\chi(M)$ in another way. Let $g : M \longrightarrow \tau(M)$ be the vector field transverse to the zero-section, as constructed in Lemma 4.15. Let $x_1, \dots, x_r \in M$ be the zeros of g. Let $\phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n$ be a chart of $\tau(M)$ over open neighbourhoods U_i of x_i in M (see Theorem 6.2, Part 1) so that ϕ_i is orientation preserving. Then the composition $h_i = (\text{proj}) \circ \phi_i \circ g$

$$U_i \xrightarrow{g} \pi^{-1}(U_i) \xrightarrow{\phi_i} U_i \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

has a regular value at $0 \in \mathbb{R}^n$ and $x_i \in h_i^{-1}(0)$ (Lemma 1.5). Since ϕ_i is orientation preserving, the orientation number $n(h_i, x_i)$ is the same as the sign of the isomorphism

$$\tau(U_i)_{x_i} \xrightarrow{d\phi_i \circ dg} \tau(U_i \times \mathbb{R}^n)_{y_i} \xrightarrow{\text{proj}} \frac{\tau(U_i \times \mathbb{R}^n)_{y_i}}{\tau(U_i \times \{0\})_{y_i}},\ y_i = \phi_i(g(x_i)),$$

which is the same as $n(g, x_i)$ (see the alternative definition of the orientation number in Definition 4.1). Therefore the Euler characteristic of M is

$$\chi(M) = \sum_i n(h_i, x_i).$$

The method of computation of $\chi(M)$ may be summarized as follows. Take a smooth vector field $f : M \longrightarrow \tau(M)$ transverse to the zero section. At each zero x_i of f, take a coordinate chart $\phi_i : U_i \longrightarrow \mathbb{R}^n$ which preserves orientation. Then the local representation of f is given by the composition

$$\phi_i(U_i) \xrightarrow{\phi_i^{-1}} U_i \xrightarrow{f} \pi^{-1}(U_i) \xrightarrow{d\phi_i} \tau(\phi_i(U_i)).$$

This is a section of the trivial bundle on $\phi_i(U_i)$, and so it defines a smooth map $g_i : \phi_i(U_i) \longrightarrow \mathbb{R}^n$ with a regular value at 0. Then, if $d_i = n(g_i, x_i)$ is the index of the vector field f at x_i, the Euler characteristic of M is

$$\chi(M) = \sum_i d_i.$$

Theorem 4.17. *$\chi(S^n) = 1 + (-1)^n$ so that it is 0 if n is odd, and it is 2 if n is even.*

Proof. Let P and $Q = -P$ be the north and the south pole of S^n. Let $U = S^n - P$ and $V = S^n - Q$, and $\phi : U \longrightarrow \mathbb{R}^n$ and $\psi : V \longrightarrow \mathbb{R}^n$ be stereographic projections from P and Q respectively (see Example 1.2(6), Part 1). Consider the atlas $\{(U, \phi), (V, \psi')\}$, where $\psi' = -\psi$, of S^n. The coordinate changes

$$\psi' \circ \phi^{-1} = \phi \circ \psi'^{-1} : \mathbb{R}^n - \{0\} \longrightarrow \mathbb{R}^n - \{0\}$$

is given by $x \mapsto -x/\|x\|^2$.

Define a section $\lambda : U \longrightarrow \pi^{-1}(U)$ by $\lambda(x) = (x, d\phi_x^{-1}(\phi(x)))$, and a section $\mu : V \longrightarrow \pi^{-1}(V)$ by $\mu(x) = (x, d\psi_x'^{-1}(\psi'(x)))$. These sections are compatible with respect to the above change of coordinates, and therefore they fit together smoothly to give a global section f of the tangent bundle (note that in the construction of f we have not used orientation of S^n). Moreover, f vanishes only at P and Q. In ϕ coordinates f corresponds to $x \mapsto x$ on $\phi(U)$, and in ψ' coordinates f corresponds to $x \mapsto -x$ on $\psi'(V)$. Since the identity map of $\mathbb{R}^n$ has degree 1, and the antipodal map has degree $(-1)^n$, $\text{Ind}_P f = 1$ and $\text{Ind}_Q f = (-1)^n$. This proves the theorem. □

Corollary 4.18 (Hairy Ball Theorem). *Every vector field on S^{2n} vanishes somewhere.*

A graphic description of this result says that a hairy ball cannot be combed continuously.

Example 4.19. Every sphere S^n of odd dimension n admits a non-zero vector field f, where, for $p = (x_1, \cdots, x_{n+1}) \in S^n$, $f(p)$ is given by

$$f(p) = (-x_2, x_1, -x_3, x_4, \cdots, -x_{n+1}, x_n).$$

This vector is orthogonal to p, and so $f(p) \in \tau(S^n)_p$.

Exercise 4.20. If M and N are compact oriented manifolds without boundaries, then show that $\chi(M \times N) = \chi(M) \cdot \chi(N)$.

Theorem 4.21. *If M is an odd dimensional compact oriented manifold without boundary, then*

$$\chi(M) = 0.$$

The converse is false as may be seen when $M = S^1 \times S^1$.

Proof. Let us compute $\chi(M)$ using a vector field f, and then using the vector field $-f$. The computations will give

$$\chi(M) = \sum_x \text{Ind}\,_x f = \sum_x \text{Ind}\,_x(-f),$$

where the summations are over all zeros x of f or $-f$. Now, if $\dim M = n$ and x is a zero of f, then

$$\text{Ind}\,_x f = (-1)^n \text{Ind}\,_x(-f).$$

This gives us the theorem, if n is odd. $\square$

5. Mod 2 Intersection Number

Let M, N, and Z be unoriented manifolds without boundary, where M is compact and Z is a closed submanifold of N such that

$$\dim\, M + \dim\, Z = \dim\, N.$$

Then, if $f : M \to N$ is a smooth map transverse to Z, $f^{-1}(Z)$ is a finite set of points. Define the **mod 2 intersection number** $I_2(f, Z)$ of f with Z to be the number of point in $f^{-1}(Z)$ modulo 2. Note that the set up is the same as before, except that now we attach the number $+1$ (or -1) to each of the points of $f^{-1}(Z)$ and then reduce the sum of these numbers mod 2.

Theorem 5.1. *Let W^{n+1} be a compact manifold with boundary M^n, and N^m be a manifold with $\partial N = \emptyset$, $m \geq n$. Let $f : M \to N$ be a smooth map, and Z be a closed subset and a submanifold of N of dimension $m - n$. If f can be extended to a smooth map $F : W \to N$, then $I_2(f, Z) = 0$.*

Proof. By Transversality homotopy theorem (Theorem 1.15), the map F is homotopic to a smooth map $G : W \to N$ such that $G \pitchfork Z$ and $\partial G \pitchfork Z$. If $g = \partial G$, then f is homotopic to g, and therefore $I_2(f, Z) = \# g^{-1}(Z) \mod 2$. But $G^{-1}(Z)$ is a compact manifold of dimension one with boundary. Therefore $\# \partial G^{-1}(Z) = \# g^{-1}(Z)$ is an even integer. $\square$

When $f : M \to N$ is a smooth map of a compact manifold M into a connected manifold N, and dim $M =$ dim N, we define the **mod 2 degree** $\deg_2(f)$ of f with any point $z \in N$ to be the mod 2 intersection numbder $I_2(f, \{z\})$. Note that this number is the same for any $z \in N$, and it is defined only when M is compact and N connected.

If dim $M = n$, and $f : M \to \mathbb{R}^{n+1}$ is a smooth map. define $u : M \to S^n$ by

$$u(x) = \frac{f(x) - z}{\|f(x) - z\|},$$

where $z \notin f(M)$. Then **mod 2 winding number** of f around z is defined to be

$$W_2(f, z) = \deg_2(u).$$

Theorem 5.2. *Let W be a compact manifold of dimension $n+1$ with boundary $\partial W = M$. Let $F : W \to \mathbb{R}^{n+1}$ be a smooth map and $f = F|M : M \to \mathbb{R}^{n+1}$. Let $z \in \mathbb{R}^{n+1}$ be a regular value of F such that $z \notin f(M)$. Then $F^{-1}(z)$ is a finite set, and*

$$W_2(f, \{z\}) = \#F^{-1}(z) \mod 2.$$

In other words, the number of times that f winds M around z is the same as the number of times F hits z, mod 2.

Proof. If $F^{-1}(z) = \emptyset$, then $f^{-1}(z) = \emptyset$ also, and therefore both F and f do not hit z. Therefore, if $\lambda : \mathbb{R}^{n+1} - \{z\} \to S^n$ is the map $a \mapsto ((a-z))/\|a - z\|$, then $\lambda \circ f : M \to S^n$ extends to $\lambda \circ F : W \to S^n$. Therefore, for any $s \in S^n$, we have by Theorem 5.1 that

$$I_2(\lambda \circ f, \{s\}) = \deg_2(\lambda \circ f) = W_2(f, \{z\}) = 0.$$

Next suppose that $F^{-1}(z) = \{x_1, \ldots, x_k\} \subset W$. Choose coordinate charts (U_i, ϕ_i) in W such that $x_i \in B_i \subset U_i$, where $\phi_i(B_i)$ is a closed ball in $\mathbb{R}^{n+1}$, and such that B_i's are disjoint from one another and from the boundary $\partial W = M$. Let

$$W' = W - \cup_{i=1}^k \operatorname{Int} B_i, \text{ and } \partial W' = M \cup_{i=1}^k \partial B_i.$$

Let $F' = F|W'$, and $f' = F'|\partial W'$. then as in the above arguments,

$$\deg_2(\lambda \circ f') = \deg_2(\lambda \circ f) + \sum_{i=1}^k \deg_2(\lambda \circ f_i) = 0.$$

This means that

$$W_2(f, \{z\}) = W_2(f_1, \{z\}) + \cdots + W_2(f_k, \{z\}) \mod 2.$$

Now, since z is a regular value of F, we can also show that each $W_2(f_i, \{z\}) = 1$. This completes the proof. □

The following theorem of Karol Borsuk and Stanislaw Ulam is an amazing theorem of topology. This implies, in particular, that at any given time there are two antipodal points on the earth's surface which have the same temperature and barometric pressure. In other words, for every smooth map $f : S^2 \to \mathbb{R}^2$, there is a pair of antipodes $\{x, -x\}$ such that $f(x) = f(-x)$.

Theorem 5.3 (**Borsul-Ulam Theorem).** *Let $f : S^n \to \mathbb{R}^{n+1}$ be a smooth map such that $0 \notin f(S^n)$, and f maps antipodal points to antipodal points, that is,*

$$f(-x) = -f(x) \quad \text{for all } x \in S^n. \tag{5.1}$$

Then $W_2(f, 0) = 1$.

The condition (5.1) may be called symmetry condition. The theorem asserts that any smooth symmetric map $f : S^n \to \mathbb{R}^{n+1} - \{0\}$ winds around the origin an odd number of times. The theorem is equivalent to the following theorem.

Theorem 5.4. *If a smooth map $g : S^n \to S^n$ maps antipodal points to antipodal points, then $\deg_2(g) = 1$.*

Proof. If $f : S^n \to \mathbb{R}^{n+1}$ is the map of Theorem 5.1, define $g : S^n \to S^n$ by

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}.$$

□

Proof of Theorem 5.3. The proof is by induction on n.

The case $n = 1$ follows easily. As described in Example 4.8, a smooth map $f : S^1 \to S^1$ can be expressed as

$$f(e^{i\theta}) = e^{ig(\theta)},$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth map such that $g(\theta + 2\pi) = g(\theta) + 2k\pi$ for all θ, where k is a fixed integer. If f is a symmetric map, then we must have $g(\theta + \pi) = g(\theta) + m\pi$, where m is an odd integer. Indeed, the symmetry condition $f(-e^{i\theta}) = -f(e^{i\theta})$ implies that $f(e^{i(\theta+\pi)}) = -f(e^{i\theta})$, or $e^{ig(\theta+\pi)} = -e^{ig(\theta)} = e^{i(g(\theta)+\pi)}$, or $g(\theta + \pi) = g(\theta) + \pi + 2k\pi$, where k is an integer. Therefore $\deg_2(f) = 1$.

Next suppose that the theorem is true for $n - 1$, and let

$$f : S^n \to \mathbb{R}^{n+1} - \{0\}$$

be a symmetric smooth map. Identify S^{n-1} with the equator of S^n by the embedding $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$. Let $g = f|S^{n-1}$.

By Sard's theorem (Theorem 1.11), choose a point $a \in S^n$ which is a regular value of both the maps

$$G = \frac{g}{\|g\|} : S^{n-1} \to S^n, \quad \text{and} \quad F = \frac{f}{\|f|} : S^n \to S^n.$$

By the symmetry condition, $-a \in S^n$ is also a regular value of both the maps.

By the preimage theorem (Theorem 4.8, Part 1), the regularity of G means that $G^{-1}(a)$ and $G^{-1}(-a)$ are manifolds of dimension one, and therefore a and $-a$ do not belong to Image G. or the line $\ell = \mathbb{R} \cdot a$ joining the points a and $-a$ does not intersect $f(S^{n-1})$. On the other hand, the regularity of F means that f is transverse to the line ℓ. To see this note that the map $F : S^n \to S^n$ is the composition $F = \lambda \circ f$, where $\lambda : \mathbb{R}^{n-1} - \{0\} \longrightarrow S^n$ is the map $\lambda(x) = x/\|x\|$, $x \in \mathbb{R}^{n+1} - \{0\}$. Then a is a regular value of F means that $F \pitchfork \{a\}$, or $(\lambda \circ f) \pitchfork \{a\}$, which is equivalent to saying that $f \pitchfork \lambda^{-1}(a)$, or $f \pitchfork \ell$, $\ell = \lambda^{-1}(a)$, by Exercise 1.12.

Now, by definition,

$$W_2(f,0) = \deg_2(F) = \# F^{-1}(a) \mod 2,$$

where the symbol $\# F^{-1}(a)$ denotes the number of points in the set $F^{-1}(a)$. Also $\#f^{-1}(\ell) = \#F^{-1}(a) + \#F^{-1}(-a) = 2\#F^{-1}(a)$, since $\#F^{-1}(a) = \#F^{-1}(-a)$, by symmetry. Therefore

$$W_2(f,0) = \frac{1}{2}\#f^{-1}(\ell) \mod 2.$$

Similar computation applies to the upper hemisphere S^n_+ of S^n

$$S^n_+ = \{x \in S^n : x_{n+1} \geq 0\}.$$

This is a manifold with boundary $\partial S^n_+ = S^{n-1}$. Let $f_+ = f|S^n_+$. Since $f(S^{n-1})$ does intersect the line ℓ, and the symmetry condition holds, $\#f^{-1}(\ell) = 2\#f_+^{-1}(\ell)$. Therefore

$$W_2(f,0) = \#f_+^{-1}(\ell) \mod 2.$$

To use the inductive hypothesis on the boundary $\partial S^n_+ = S^{n-1}$, we need to adjust the dimension of the range of g. For this purpose, let V be the n dimensional space which is the orthogonal complement of the line ℓ in $\mathbb{R}^{n+1}$, and $\pi : \mathbb{R}^{n+1} \to V$ be the orthogonal projection. Since g is symmetric, and π is linear, the composition $\pi \circ g : S^{n-1} \to V \equiv \mathbb{R}^n$ is symmetric. Moreover, $\pi \circ g$ is never zero, since g never intersects $\pi^{-1}(0) = \ell$. Therefore we have by the inductive hypothesis,

$$W_2(\pi \circ g, 0) = 1.$$

Now, since $f_+ \pitchfork \ell$, $\pi \circ f_+ : S^n_+ \to V$, is transverse to $\{0\}$. Therefore, by Theorem 5.2,

$$W_2(\pi \circ g, 0) = \# (\pi \circ f_+)^{-1}(0).$$

But $(\pi \circ f_+)^{-1}(0) = f_+^{-1}(\ell)$. Therefore

$$W_2(f,0) = \# f_+^{-1}(\ell) = W_2(\pi \circ g, 0) = 1 \mod 2.$$

This completes the proof.

Theorem 5.5 (Jordan-Brouwer Separation Theorem). *Let M be a compact connected manifold of dimension $n-1$ in $\mathbb{R}^n$. Then the complement $\mathbb{R}^n - M$ consists of two connected open sets, the inside M_0 and the outside M_1. Moreover, $\overline{M}_0$ is a compact manifold with boundary $\partial\,\overline{M}_0 = M$.*

The proof is similar to the last theorem and is left as an exercise (see Guillemin and Pollack, Differential Topology, p. 89 in case of difficulty).