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Lecture on Differential Topology Part 4

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1. Vector Bundle

Definition 1.1. A **vector bundle** of dimension n is a triple (E, M, π) consisting of a pair of manifolds E and M connected by a smooth surjective map $\pi : E \longrightarrow M$ satisfying the following conditions.

VB1. For each $x \in M$, the inverse image $E_x = \pi^{-1}(x)$ is an n-dimensional vector space over $\mathbb{R}$,

VB2. For each $x \in M$, there is an open neighbourhood U of x and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that

(i) $p_1 \circ \phi = \pi$, where $p_1 : U \times \mathbb{R}^n \to U$ is the projection onto the first factor.

(ii) for each $y \in U$, the map $\phi_y : \pi^{-1}(y) \to \mathbb{R}^n$, defined by the composition

$$\pi^{-1}(y) \xrightarrow{\phi} \{y\} \times \mathbb{R}^n \xrightarrow{p_2} \mathbb{R}^n,$$

where p_2 is the projection onto the second factor, is a linear isomorphism.

Note that locally π is the composition of a diffeomorphism ϕ followed by a submersion p_1, therefore π is a submersion.

The vector bundle is also denoted by the map $\pi : E \longrightarrow M$ (and sometimes E itself is called the vector bundle, by an abuse of language). The manifold E is called the **total space** of the bundle, the manifold M its **base space**, and the map π its **projection**. The inverse image $E_x = \pi^{-1}(x), x \in M$, is called the **fibre over** x. The condition VB2 is called the **local triviality**; the pair (U, ϕ) is called a **vector bundle chart** with domain U, and U is called a **trivializing open set**. A collection $\Phi = \{(U_i, \phi_i)\}$ of charts, whose domains cover M, is called a **vector bundle atlas** if whenever (U_i, ϕ_i) and (U_j, ϕ_j) are in Φ and $x \in U_i \cap U_j$ the diffeomorphism $(\phi_j)_x^{-1} \circ (\phi_i)_x : \mathbb{R}^n \to \mathbb{R}^n$ is linear.

Notice that the dimension of a vector bundle E is actually the dimension of its fibre, and is not the dimension of E as a manifold. The function $M \longrightarrow \mathbb{R}$ given by $x \mapsto \dim E_x$ is a locally constant function, and therefore it is a constant on each component of M. If the function is constant on the whole of M, then the common value $\dim E_x$, for all $x \in M$, is the dimension of the vector bundle E.

Example 1.2. If V is a finite dimensional vector space over $\mathbb{R}$, then the projection $\pi : M \times V \longrightarrow M$ is a vector bundle. This is called a **product bundle**.

Example 1.3. If $\pi_1 : E_1 \longrightarrow M$ and $\pi_2 : E_2 \longrightarrow M$ are vector bundles over M of dimension n and m respectively, let

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 \mid \pi_1(v_1) = \pi_2(v_2)\}.$$

Then $\pi : E_1 \oplus E_2 \longrightarrow M$ given by $\pi(v_1, v_2) = \pi_1(v_1) = \pi_2(v_2)$ is a vector bundle of dimension $n + m$, whose fibre over $x \in M$ is the direct sum $\pi_1^{-1}(x) \oplus \pi_2^{-1}(x)$. If

(U, ϕ_1) and (U, ϕ_2) are charts for E_1 and E_2 respectively over a common trivializing open set U, then a chart (U, ψ) for $E_1 \oplus E_2$ is obtained by setting

$$\psi_x = (\phi_1)_x \oplus (\phi_2)_x : (E_1)_x \oplus (E_2)_x \to \mathbb{R}^n \oplus \mathbb{R}^m, \quad x \in U.$$

The bundle $E_1 \oplus E_2$ is call the **Whitney sum** of E_1 and E_2.

Example 1.4. The tangent bundle $\tau(M)$ of a manifold M is a vector bundle with dimension equal to $\dim M$. The charts constructed in Theorem 6.2, Part 1, to show that $\tau(M)$ is a manifold are actually vector bundle charts.

Definition 1.5. If (E, M, π) and (E', M', π') are vector bundles, then a **morphism** $(f, g) : (E, M, \pi) \longrightarrow (E', M', \pi')$ consists of a pair of smooth maps $f : E \longrightarrow E'$ and $g : M \longrightarrow M'$ such that

(i) the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & E' \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ M & \xrightarrow[\ g\ ]{} & M' \end{array}$$

(ii) the restriction of f to each fibre

$$f_x = f|E_x : E_x \longrightarrow E'_{g(x)}, \qquad x \in M,$$

is a linear map.

A pair $(\mathrm{Id}_E, \mathrm{Id}_M)$ is the identity morphism. The composition of two morphisms (f, g) and (f', g') is defined to be $(f' \circ f, g' \circ g)$, and this is again a morphism.

A morphism $(f, \mathrm{Id}_M) : (E, M, \pi) \longrightarrow (E', M, \pi')$ over the same base space M is called a **homomorphism**, and sometimes it is denoted simply by $f : E \longrightarrow E'$. It is called a **monomorphism, epimorphism**, or **isomorphism** according to whether each f_x is a monomorphism, epimorphism, or isomorphism. It is called a **bundle equivalence** if f is an isomorphism and a diffeomorphism. In this case, each $f|E_x$ is a linear isomorphism, and its inverse is the restriction to E'_x of f^{-1}.

Lemma 1.6. *A homomorphism $f : (E, M, \pi) \longrightarrow (E', M, \pi')$ is a bundle equivalence if and only if f is an isomorphism.*

Proof. It suffices to show that a smooth map $f : E \longrightarrow E'$ over M which maps each fibre isomorphically onto a fibre, is a diffeomorphism. Define a map $g : E' \longrightarrow E$ by $g(\alpha') = f_x^{-1}(\alpha')$, where $\alpha' \in E'$, and $\pi'(\alpha') = x$. Then f will be a diffeomorphism, if we show that g is a smooth map.

Let $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ and $\phi' : (\pi')^{-1}(U) \to U \times \mathbb{R}^n$ be vector bundle charts for E and E' respectively corresponding to a common trivializing open set U in M. Then $\phi' \circ f \circ \phi^{-1}$ is of the form $(x, v) \mapsto (x, h(x)v)$, where $h : U \longrightarrow L(\mathbb{R}^n, \mathbb{R}^n)$, the range of h is the space of linear maps $\mathbb{R}^n \to \mathbb{R}^n$. Thus the map $f : \pi^{-1}(U) \longrightarrow (\pi')^{-1}(U)$ over U is smooth if and only if h is smooth. Also f is an isomorphism on each fibre if and only if $\mathrm{Im}\ h \subset GL(n, \mathbb{R})$. Since the map $GL(n, \mathbb{R}) \longrightarrow GL(n, \mathbb{R})$ given by $\lambda \mapsto \lambda^{-1}$ is smooth, the map $h^{-1} : U \longrightarrow GL(n, \mathbb{R})$, given by $h^{-1}(x) = h(x)^{-1}$, is also smooth, if h is so. Using these facts, we find

$$f|\pi^{-1}(U) \text{ smooth } \Rightarrow \phi' \circ f \circ \phi^{-1} \text{ smooth } \Rightarrow h \text{ smooth } \Rightarrow$$

$$h^{-1} \text{ smooth } \Rightarrow \phi \circ g \circ (\phi')^{-1} \text{ smooth } \Rightarrow g|(\pi')^{-1}(U) \text{ smooth }.$$

Since this is true for any common trivializing open set U, g is a smooth map. □

A vector bundle is called **trivial** if it is equivalent to a product bundle.

Implicit in the above definition of vector bundle is an important role of the general linear group $GL(n, \mathbb{R})$, which appears in a transition law (the equation (1.1) below) between vector bundle charts. This may be described in the following way. The definition of a vector bundle $\pi : E \longrightarrow M$ guarantees the existence of a trivializing covering $\{U_i\}_{i\in\Lambda}$ of M (Λ = index set) and the vector bundle charts $\phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n$ satisfying VB2. Then, for any $i, j \in \Lambda$ with $U_i \cap U_j \neq \emptyset$, the diffeomorphism

$$\phi_j \circ \phi_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^n \longrightarrow (U_i \cap U_j) \times \mathbb{R}^n$$

must be of the form

$$\phi_j \circ \phi_i^{-1}(x, v) = (x, g_{ij}(x)v) \tag{1.1}$$

for a unique smooth map $g_{ij} : U_i \cap U_j \longrightarrow GL(n, \mathbb{R})$. In fact, if $\phi_{ix} : \pi^{-1}(x) \longrightarrow \mathbb{R}^n$ is the map $\phi_{ix}(v) = \phi_i(x, v)$, then

$$g_{ij}(x) = \phi_{jx} \circ \phi_{ix}^{-1}$$

is a linear isomorphism $\mathbb{R}^n \longrightarrow \mathbb{R}^n$. The family of maps $\{g_{ij}\}$ is called a **cocycle**. They satisfy the following condition

$$g_{ij}(x) \cdot g_{jk}(x) = g_{ik}(x), \qquad x \in U_i \cap U_j \cap U_k.$$

Putting $i = j = k$, and then $k = i$ in this condition, we get

$$g_{ii}(x) = \text{ id, for all } x \in U_i, \text{ and } g_{ij}(x) = (g_{ji}(x))^{-1} \text{ for all } x \in U_i \cap U_j.$$

Definition 1.7. A **section** of a vector bundle
$p : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{Id}_M$.

For example, A section of the tangent bundle $\tau(M)$ is a vector field (Definition 6.5, Part 1).

The section $i : M \to E$, which maps $x \in M$ to the zero vector $0_p \in E_x$, is called the **zero section**. Then i is an embedding. We often identify M with the $i(M) \subset E$, and call $i(M)$ the zero section.

Definition 1.8. A **metric** on a vector bundle $E \longrightarrow M$ is a smooth map which assigns to each $x \in M$ a positive definite symmetric bilinear form or inner product

$$\langle\ ,\ \rangle_x : E_x \times E_x \longrightarrow \mathbb{R}.$$

It can be shown that the collection of all metrics on E form a vector bundle $S^2(E)$ over M, whose fibre over $x \in M$ is the vector space of all inner products on the fibre E_x. Then a metric on E will be a smooth section of the vector bundle $S^2(E)$.

A metric on the tangent bundle $\tau(M)$ is a Riemannian metric on M, which we have discussed already in Part 2, §2.

The standard metric $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ defines a metric on the product bundle $M \times \mathbb{R}^n$ by

$$\langle (x, v), (x, w) \rangle_x = \langle v, w \rangle.$$

Theorem 1.9. *Any vector bundle E over M admits a metric.*

Proof. The proof is similar to that of Theorem 2.3, Part 2. One has to take metrics on the trivial bundles $E|U_i$, for some trivializing open covering $\{U_i\}$ of M, and then splice them together using a smooth partition of unity. $\square$

Let $\pi : E \to N$ be a vector bundle over N, and $f : M \to N$ be a smooth map. Then the **pull-back** (or **induced bundle**) of E by f is the vector bundle $\pi' : f^*(E) \to M$ over M , where $f^*(E) = \{(x, v) \in M \times E : f(x) = \pi(v)\}$, and $\pi'(x, v) = x$.

If $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ is a chart for E, and $V = f^{-1}(U) \subset M$, then $\psi : {\pi'}^{-1}(V) \to V \times \mathbb{R}^n$ given by $\psi(x, \alpha) = (x, p_2 \circ \phi(\alpha))$, where $x \in V, \alpha \in \pi^{-1}(V)$ so that $f(x) = \pi(\alpha)$, is a chart for $f^*(E)$. Note that the inverse of ψ is given by $\psi^{-1}(x, v) = (x, \phi^{-1}(f(x), v))$.

There is a morphism $(\widetilde{f}, f) : f^*(E) \to E$ given by $\widetilde{f}(x, v) = v$ such that each $\widetilde{f}_x : f^*(E)_x \to E_{f(x)}$ is an isomorphism

$$\begin{array}{ccc} f^*(E) & \xrightarrow{\widetilde{f}} & E \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow[f]{} & N \end{array}$$

Pull-backs satisfy the following properties :

(i) $\mathrm{Id}^*(E) = E$,

(ii) $(g \circ f)^*(E) \simeq f^*(g^*(E))$,

(iii) $f^*(E_1 \oplus E_2) \simeq f^*(E_1) \oplus f^*(E_2)$.

Lemma 1.10. *Any vector bundle morphism*

$$(g, f) : (E_1, M_1, \pi_1) \longrightarrow (E_2, M_2, \pi_2)$$

can be factored as $(g, f) = (k, f) \circ (h, \mathrm{Id})$, $g = k \circ h$,

$$\begin{array}{ccccc} E_1 & \xrightarrow{h} & f^*(E_2) & \xrightarrow{k} & E_2 \\ \big\downarrow{\scriptstyle \pi_1} & & {\scriptstyle \pi}\big\downarrow & & {\scriptstyle \pi_2}\big\downarrow \\ M_1 & \xrightarrow{\mathrm{Id}} & M_1 & \xrightarrow{f} & M_2 \end{array}$$

where (h, Id) is a homomorphism and (k, f) is a morphism such that k maps each fibre isomorphically onto a fibre.

Proof. Define $k : f^*(E_2) \longrightarrow E_2$ by $k(x, \alpha) = \alpha$, and $h : E_1 \longrightarrow M_1 \times E_2$ by $h(\alpha) = (\pi_1(\alpha), g(\alpha))$. Since $f \circ \pi_1 = \pi_2 \circ g$, Im $h \subset f^*(E_2)$. Then h is linear on each fibre, and $g = k \circ h$. $\square$

Definition 1.11. A vector bundle $\pi' : E' \to M$ is a **subbundle** of a vector bundle $\pi : E \to M$ if E' is a submanifold of E and $\pi' = \pi|E'$.

Note that if E' is a subbundle of E, then the inclusion $f : E' \to E$ is a monomorphism. The converse is also true.

Lemma 1.12. *If E and E' are vector bundles over M and $f : E' \longrightarrow E$ is a monomorphism, then $f(E')$ is a subbundle of E, and $f : E' \longrightarrow f(E')$ is a bundle equivalence.*

Proof. It is sufficient to prove that each $x \in M$ has an open neighbourhood U on which $f(E')$ is a subbundle of E. Therefore we may suppose that E and E' are product bundles. Let $E = M \times \mathbb{R}^n$, and, for $x \in M$, let V_x be a subspace complementary to $f(E'_x)$ in $\mathbb{R}^n$. Then $F = M \times V_x$ is a subbundle of E. Define $g : E' \oplus F \longrightarrow E$ by $g(u, v) = f(u) + i(v)$ where $i : F \longrightarrow E$ is the inclusion. Then g_x is an isomorphism. Therefore there is an open neighbourhood U of x in M such that $g|U$ is an isomorphism, and a hence a diffeomorphism, by Lemma 1.6. Now, E' is a subbundle of $E' \oplus F$. Therefore $g(E') = f(E')$ is a subbundle of $g(E' \oplus F) = E$ on U. The second part also follows from Lemma 1.6 □

Remarks 1.13. The proof shows that

1. if $f : E' \longrightarrow E$ is a homomorphism, then the set of points $x \in M$ for which f_x is a monomorphism is an open set of M,
2. locally a subbundle E' of a bundle E is a direct summand of E.

Definition 1.14. If E' is a subbundle of E, then the **quotient bundle** E/E' (of E modulo E') is the union of all vector spaces E_x/E'_x with the quotient topology.

Note that since E' is locally a direct summand in E, E/E' is locally trivial, and hence it is a vector bundle.

Proposition 1.15. *Let $(f, g) : (E', M', \pi') \longrightarrow (E, M, \pi)$ be a vector bundle morphism of constant rank, that is, f_x has constant rank for all $x \in M'$, Then*

(i) $\ker f = \cup_x \ker f_x$ *is a subbundle of E',*
(ii) $\mathrm{Im} f = \cup_x \mathrm{Im} f_x$ *is a subbundle of E.*
(iii) $\mathrm{Coker} f = \cup_x \mathrm{Coker} f_x$ *is a quotient bundle of E.*

Proof. The assertion (ii) implies (iii). We shall first prove (ii). The problem is local, and therefore we assume that $E' = M \times \mathbb{R}^n$. Let $x \in M$, and V_x be a complement of $\ker f_x$ in $\mathbb{R}^n$. Then $F = M \times V_x$ is a subbundle of E', and the homomorphism $g = f \circ i : F \longrightarrow E$ (i = inclusion) is such that g_x is a monomorphism. Therefore g is a monomorphism in some open neighbourhood U of x. Therefore $g(F)|U$ is a subbundle of $E|U$, by Lemma 1.12. Now $g(F) \subset f(E')$, and, since $\dim f(E'_y)$ is constant for all $y \in M$, we have, for all $y \in U$,

$$\dim g(F_y) = \dim g(F_x) = \dim f(E'_x) = \dim f(E'_y).$$

Therefore $g(F)|U = f(E')|U$, and $f(E')$ is a subbundle of E.

We next prove (i). Note that a homomorphism $f : E' \longrightarrow E$ is a monomorphism if and only if its dual $f^* : E^* \longrightarrow E'^*$ is an epimorphism. Also f has constant rank implies that f^* has constant rank. Therefore, since $E'^* \longrightarrow E'^*/f^*(E^*) = \mathrm{Coker} f^*$ is an epimorphism, $(\mathrm{Coker} f^*)^* \longrightarrow (E'^*)^*$ is a monomorphism. Now

$$\mathrm{Im} f_x^* = \{\alpha : E'_x \longrightarrow \mathbb{R} \mid \ker f_x \subset \ker\ \alpha\}.$$

Therefore $\mathrm{Coker} f_x^*$ can be identified with the subspace of elements $\alpha \in {E'}_x^*$ such that, for some non-zero $v \in E'_x$, $f_x(v) = 0$ but $\alpha(v) \neq 0$. Then there is an isomorphism

$$\eta : \ker f_x \longrightarrow L(\mathrm{Coker} f_x^*, \mathbb{R}) = (\mathrm{Coker} f_x^*)^*$$

given by $\eta(v)(\alpha) = \alpha(v)$. In fact, we have for each $x \in M$ a natural commutative diagram

$$\begin{array}{ccc} \ker f_x & \longrightarrow & E'_x \\ \eta \downarrow & & \downarrow \eta \\ (\mathrm{Coker} f_x^*)^* & \longrightarrow & ({E'}_x^*)^* \end{array}$$

where the vertical arrows η are isomorphisms. Thereore

$$\ker f \simeq (\mathrm{Coker} f^*)^*,$$

and, by Lemma 1.12, is a subbundle of E'. □

Definition 1.16. A sequence of vector bundles $\{E_i\}$ over M connected by homomorphisms $\{f_i\}$ over Id_M

$$\cdots \longrightarrow E_{i-1} \xrightarrow{f_{i-1}} E_i \xrightarrow{f_i} E_{i+1} \longrightarrow \cdots$$

is called an **exact sequence** over M if for each $x \in M$ we have

$$\mathrm{Im}(f_{i-1})_x = \ker(f_i)_x, \text{ for all } i.$$

In particular, a five-term exact sequence over M:

$$0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \longrightarrow 0,$$

where 0 denotes the vector bundle of dimension 0, is called a **short exact sequence**. Here exactness means that f is a monomorphism, g is an epimorphism, and $\mathrm{Im} f = \ker\ g$.

The bundle E'' of the above short exact sequence is called the **quotient bundle** of the monomorphism f.

For the justification of the terminology note that if E' is a subbundle of E with inclusion homomorphism i, then we have the short exact sequence

$$0 \longrightarrow E' \xrightarrow{i} E \xrightarrow{p} E/E' \longrightarrow 0,$$

where p is the quotient homomorphism. Moreover, every monomorphism has a quotient bundle

$$0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{p} E/f(E') \longrightarrow 0,$$

and it is unique up to isomorphism, as the following lemma shows.

Lemma 1.17. *If E_1 and E_2 are two quotient bundles of a monomorphism $f : E' \longrightarrow E$, then there is a unique isomorphism $h : E_1 \longrightarrow E_2$ so that the following diagram commutes.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \xrightarrow{f} & E & \xrightarrow{g_1} & E_1 & \longrightarrow & 0 \\ & & \mathrm{Id} \downarrow & & \mathrm{Id} \downarrow & & h \downarrow & & \\ 0 & \longrightarrow & E' & \xrightarrow{f} & E & \xrightarrow{g_2} & E_2 & \longrightarrow & 0 \end{array}$$

Proof. The map h is defined as follows. Let $v \in E_1$. Then, by exactness, there is a $u \in E$ such that $g_1(u) = v$. Then set $h(v) = g_2(u)$. If u' is another element of E for which $g_1(u') = v$ also, then $u - u' \in \ker g_1 = \mathrm{Im} f$, so there is a $w \in E'$ such that $f(w) = u - u'$. This means that $g_2 \circ f(w) = g_2(u - u') = 0$, or $g_2(u) = g_2(u')$, showing that h is well defined. It is easily checked that h is actually an isomorphism. $\square$

Exercise 1.18. Show that given an epimorphism $g : E \longrightarrow E''$, there is a unique bundle E' that fits into an exact sequence

$$0 \longrightarrow E' \longrightarrow E \xrightarrow{g} E'' \longrightarrow 0.$$

Lemma 1.19. *Given a short exact sequence over a manifold M*

$$0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \longrightarrow 0,$$

there is an equivalence $\phi : E \longrightarrow E' \oplus E''$ such that $\phi \circ f$ is the natural inclusion $i : E' \longrightarrow E' \oplus E''$, and $g \circ \phi^{-1}$ is the natural projection $p : E' \oplus E'' \longrightarrow E''$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E' & \xrightarrow{f} & E & \xrightarrow{g_1} & E'' & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & E' & \xrightarrow{i} & E \oplus E'' & \xrightarrow{p} & E'' & \longrightarrow & 0
\end{array}$$

Proof. Equip E with a fibrewise metric $\langle\ ,\ \rangle_x, x \in M$. Let $f(E')_x^{\perp}$ be the subspace of E_x orthogonal to the subspace $f(E')_x$

$$f(E')_x^{\perp} = \{v \in E_x \mid \langle v, w \rangle_x = 0 \text{ for all } w \in f(E')_x\}.$$

Let $f(E')^{\perp}$ be the union of all $f(E')_x^{\perp}$, $x \in M$. Then $f(E')^{\perp}$ is a vector bundle, since it is the kernel of a homomorphism of constant rank, which is the orthogonal projection of E onto $f(E')$. We have then $E = f(E') \oplus f(E')^{\perp}$, and a short exact sequence

$$0 \longrightarrow f(E') \xrightarrow{i} E \xrightarrow{p} f(E')^{\perp} \longrightarrow 0.$$

Thus there are two quotient bundles E'' and $f(E')^{\perp}$ of the monomorphism $f : E' \longrightarrow E$; therefore they are equivalent by Lemma 1.17.

If h is the equivalence $f(E')^{\perp} \longrightarrow E''$, then the required equivalence ϕ is given by

$$E = f(E') \oplus f(E')^{\perp} \xrightarrow{f \oplus h} E' \oplus E''.$$

$\square$

Remark 1.20. The proof of the lemma contains the definition of orthogonal bundle, which may be singled out as follows. If E' is a subbundle of a bundle E with a metric, then the **orthogonal complement** of E' in E is the bundle ${E'}^{\perp}$ whose fibre over $x \in M$ is given by

$${E'}_x^{\perp} = \{v \in E_x \mid \langle v, w \rangle_x = 0, w \in E'_x\}.$$

2. Tubular neighbourhood theorem

Let M be a submanifold of N. Then a **tubular neighbourhood** of M in N consists of a vector bundle $\pi : E \to M$ over M, and a diffeomorphism θ of an open neighbourhood Z of the zero section $i(M)$ in E onto an open neighbourhood U of M in N

$$\theta : Z \to N$$

such that $\theta \circ i$ is the inclusion $M \subset N$.

Then Z is called the tubular neighbourhood of M in E, or $U = \theta(Z)$ is called the tubular neighbourhood of M in N.

Theorem 2.1. *Let M be a submanifold of an Euclidean space $\mathbb{R}^m$ with $\partial M = \emptyset$. Then M has a tubular neighbourhood in $\mathbb{R}^m$.*

Proof. We shall prove the theorem when M is compact. The general case will be considered at the end.

We give M the Riemannian metric induced for $\mathbb{R}^m$. Then the normal space $\nu(M)_x$ of M at $x \in M$ is the orthogonal complement of the tangent space $\tau(M)_x$ in $\tau(\mathbb{R}^m)_x = \mathbb{R}^m$

$$\nu(M)_x = \{(x, v) \in M \times \mathbb{R}^m : v \perp \tau(M)_x\}.$$

The normal bundle $\nu(M)$ of M is the union of all normal spaces $\nu(M) = \cup_{x \in M} \nu(M)_x$. Let $\pi : \nu(M) \to M$ be the projection map $\pi(x, v) = x$. Then $\pi : \nu(M) \to M$ is a vector bundle of rank $m - n$.

Let $\epsilon > 0$ be a real number. Consider a neighbourhood of M in $\nu(M)$ defined by

$$\nu(M, \epsilon) = \{(x, v) \in \nu(M) : \|v\| < \epsilon\}.$$

Let $\theta : \nu(M) \to \mathbb{R}^m$ be the smooth map $\theta(x, v) = x + v$. We shall show that there is an $\epsilon > 0$ such that the map $\theta : \nu(M, \epsilon) \to \mathbb{R}^m$ is a diffeomorphism onto the open neighbourhood

$$\{y \in \mathbb{R}^m : \text{dist}\,(M, y) < \epsilon\}$$

of M in $\mathbb{R}^m$.

First note that there is a canonical splitting

$$\tau(\nu(M))_{(x,0)} = \tau(M)_x \oplus \nu(M)_x.$$

Let us compute the differential $d\theta$ at $(x, 0) \in \nu(M)$. Since $\theta(x, v) = x + v$ is a translation for fixed x and variable v, $d\theta_x : \nu(M)_x \to \tau(\mathbb{R}^m)_x$ is the standard inclusion map. Also $d\theta : \tau(M) \to \tau(\mathbb{R}^m)$ is just the differential of the inclusion map $M \subset \mathbb{R}^m$. Therefore $d\theta_x : \tau(M)_x \to \tau(\mathbb{R}^m)_x$ is the inclusion of $\tau(M)_x$ in $\tau(\mathbb{R}^m)_x = \mathbb{R}^m$. Therefore

$$d\theta_x : \mathbb{R}^m = \tau(\mathbb{R}^m)_x = \tau(M)_x \oplus \nu(M)_x \to \mathbb{R}^m$$

is the identity map. Therefore $d\theta$ is an isomorphism at $(x, 0)$ for every $x \in M$, and so θ is a diffeomorphism on some neighbourhood of $(x, 0)$. Therefore $(df)_{x,v}$ is an isomorphism for any x and for $\|v\|$ small.

Since M is compact, there is a $\delta > 0$ such that $d\theta$ is an isomorphism at all points of $\nu(M, \delta)$. Therefore $\theta(\nu(M, \delta)) \to \mathbb{R}^m$ is a local diffeomorphism. We shall show that θ is one-one on $\nu(M, \epsilon)$ for some $0 < \epsilon \leq \delta$.

Suppose that θ is not injective on $\nu(M,\epsilon)$ for any $\epsilon > 0$. Then there exist sequences $(x_i, v_i) \neq (y_i, w_i)$ in $\nu(M)$ such that $\|v_i\| \to 0$, $\|w_i\| \to 0$, and $\theta(x_i, v_i) = \theta(y_i, w_i)$. Since M is compact and metrizable, there exist subsequences such that $x_i \to x$ and $y_i \to y$ (by reindexing). Then

$$\theta(x_i, v_i) \to \theta(x, 0) = x \quad \text{and} \quad \theta(y_i, w_i) \to \theta(y, 0) = y,$$

so that $x = y$. But then, for large i, both (x_i, v_i) and (y_i, w_i) are close to $(x, 0)$. This is a contradiction, since θ is injective near $(x, 0)$. Thus θ must be injective on some $\nu(M,\epsilon)$.

To complete the proof, we must show that $\theta(\nu(M,\epsilon)) = \{y : \operatorname{dist}(y, M) < \epsilon\}$. The part "$\subset$" is clear. So suppose that y is such that $d(y, M) < \epsilon$ and let $x \in M$ be such that $d(y, x)$ is minimum (and hence $< \epsilon$). Then the vector $y - x$ normal vector at the point x, and so y does lie in $\theta(\nu(M,\epsilon))$. This completes the proof of the theorem when M is compact.

For a general manifold M the proof may be completed using the following lemma, whose proof may be found in Godement, Théorie des Faisceaux, page 150.

Lemma. Let $f : M \to N$ be a smooth map, and $A \subset M$, $B \subset N$ be submanifolds such that $df_p : \tau(M)_p \to \tau(N)_{f(p)}$ is an isomorphism for every $p \in A$, and $f|A : A \to B$ is a diffeomorphism. Then there is an open neighbourhood V of A in M such that $f(V)$ is an open neighbourhood of B in N, and $f|V$ is a diffeomorphim. $\square$

In general, the normal bundle of a submanifold M in N is defined to be the quotient bundle of $\tau(N)|M$ modulo $\tau(M)$

$$\nu(M) = (\tau(N)|M)/\tau(M).$$

In view of Lemma 1.17, the bundle $\nu(M)$ is equivalent to the orthogonal bundle $\tau(M)^{\perp}$ of $\tau(M)$ in $\tau(N)$ with respect to a Riemannian metric on N. Then the fibre $\nu(M)_x$ is the normal space considered in Theorem 1.1. Note that the present definition does not depend on the choice of the Riemannian metric

We have a short exact sequence of vector bundles over M

$$0 \to \tau(M) \to \tau(N)|M \to \nu(M) \to 0,$$

and a splitting

$$\tau(N)|M = \tau(M) \oplus \nu(M).$$

Theorem 2.2. *If M is a submanifold of N, and $\partial M = \partial N = \emptyset$, then M has a tubular neighbourhood in N.*

Proof. We may assume that N is a submanifold of some $\mathbb{R}^m$. Then N has a tubular neighbourhood Z in $\mathbb{R}^m$, and a smooth retraction $r : Z \to N$. We give M the Riemannian metric induced from N, and let $\nu(M)$ be the normal bundle of M in N. Then

$$\nu(M) \subset \tau(N)|M \subset \tau(\mathbb{R}^m)|M = M \times \mathbb{R}^m,$$

and each fibre $\nu(M)_x$ is contained in $\{x\} \times \mathbb{R}^m$.

For each $x \in M$, let

$$U_x = \{(x, v) \in \nu(M)_x : x + v \in Z\},$$

and $U = \cup_{x \in M} U_x$. Then $U = \phi^{-1}(Z)$, where $\phi : \nu(M) \to \mathbb{R}^m$ is the map $\phi(x, v) = x + v, \;\; v \in \nu(M)_x$, and so U is open in $\nu(M)$. Clearly,- the map $\theta = r \circ \phi : U \to N$ gives a tubular neighbouthood for M in N. □

3. Collar neighbourhoods

et M be a manifold with boundary. The boundary ∂M cannot have tubular neighbourhood. However, it has a collar neighbourhood, which is a knid of 'half-tubular' neighbourhood. The precise definition is as follows.

Definition 3.1. A **collar neighbourhood** of ∂M in M is an embedding

$$\phi : \partial M \times [0, \infty) \longrightarrow M$$

such that $\phi(x, 0) = x$.

Theorem 3.2 (**Existence of collar neighbourhood).** *There exists a collar neighbourhood of ∂M in M.*

At a boundary point, we have two kinds of tangent vectors to M, inward- and outward-pointing tangent vectors. precisely, in terms of local coordinates, a tangent vector to M has the form $\sum_i \lambda_i \partial/\partial x_i$. A tangent vector is inward- (resp. outward-) pointing if $\lambda_1 > 0$ (resp. $\lambda_1 < 0$). If $\lambda_1 = 0$, then it is a tangent to the boundary ∂M. Note that such vectors form the image of the inclusion $di : \tau(\partial M) \to \tau(M)$, where $i : \partial M \subset M$, and so they are tangent to ∂M.

In order to show the existence of collar neighbourhood in the sprit of Theorem 2.2, all we need is to identify $\partial M \times [0, -\infty)$ with the set of inward-pointing normal vectors to ∂M. Thye identification is possible, because there is only one such normal direction at each point of ∂M, and so an inward-pointing normal vector is determined by its length. Then the previous arguments of Theorem 2.2 (along with some differential geometry) carry over to this present case.

This proof may seem rather involved. Let us therefore try an alternative approach to the proof using differential equations in a straightforward way without having recourse to geodesics and the exponential map.

This proof will follow after the next two lemmas (Lemma 3.3 and Lemma 3.4).

Lemma 3.3. *Let M be a manifold without boundary, and X_0 be the constant unit vector field on $M \times \mathbb{R}$ whose value $(X_0)_{(x,\ t)}$ at any point $(x, t) \in M \times \mathbb{R}$ is tangent to the curve $t \mapsto (x, t)$ at that point. Then, for any smooth vector field X on M, there is a positive smooth function $\epsilon : M \longrightarrow \mathbb{R}$, and a unique smooth map $f : W(\epsilon) \longrightarrow M$, where $W(\epsilon) = \{(x, t) \in M \times \mathbb{R} \mid |t| < \epsilon(x)\}$, such that*

(1) $f(x, 0) = x$,
(2) $df_{(x,t)}((X_0)_{(x,\ t)}) = X_{f(x,\ t)}$.

The converse is trivially true: given f with conditions (1) and (2), the vector field X may be defined by (2).

Proof. Let X be a vector field on M. Consider an atlas $\{(U_\alpha, \phi_\alpha)\}$ for M. On a coordinate neighbourhood U_α with local coordinates $\phi_\alpha = (x_1, \ldots, x_n)$, the vector field $X_\alpha = X|U_\alpha$ has representation $X_\alpha = a_1 \cdot \partial/\partial x_1 + \cdots + a_n \cdot \partial/\partial x_n$, where a_i :

$U \longrightarrow \mathbb{R}$ are smooth functions. The vector field X_0 on $M\times\mathbb{R}$ has $n+1$ components $(0,\dots,0,1)$ (with 1 in the $(n+1)$-th slot) in any coordinate neighbourhood of $M\times\mathbb{R}$. Then the condition (2) gives a system of differential equations

$$\frac{\partial f_i}{\partial t}(x,t) = a_i(f_1(x,t),\dots,f_n(x,t)),$$

where $f_i : U_\alpha \times \mathbb{R} \longrightarrow \mathbb{R}$ are the components of f, with an initial condition corresponding to (1) at $t = 0$.

Solving the system of equations, we get a unique solution

$$f_\alpha : V_\alpha \times (-\epsilon_\alpha, \epsilon_\alpha) \longrightarrow U_\alpha.$$

where $V_\alpha = \phi_\alpha^{-1}(B)$ for some small open n-ball B in $\mathbb{R}^n$, and ϵ_α is some small positive number.

We may suppose that $V_\alpha \subseteq U_\alpha$. Then carrying out the construction for every α, we have $f_\alpha = f_\beta$ on $V_\alpha \cap V_\beta$ by the uniqueness. Also we may get a positive smooth function $\epsilon : M \longrightarrow \mathbb{R}$ as $\epsilon = \sum_\alpha \lambda_\alpha \epsilon_\alpha$, by gluing the constant functions ϵ_a on V_α by a partition of unity $\{\lambda_\alpha\}$ subordinate to the covering $\{V_\alpha\}$. Therefore we may define the required map $f : W(\epsilon) \longrightarrow M$ by $f|V_\alpha = f_\alpha$. □

The proof of the lemma will break down if M has boundary. In this case, if $x \in \partial U_\alpha$, then the solution f_α may not lie in U_α, as its first component may not be ≥ 0. However, if the vector field $X_\alpha = X|U_\alpha$ is inward pointing at any point of ∂U_α, that is, if its first component a_1 is positive, then the solution $f_1(x,t)$ will be positive for small values of $t \geq 0$. To take care of this situation, we need to construct a vector field X on M such that, for each $x \in \partial M$, the vector X_x is the inward pointing.

So, suppose that M is a manifold with boundary, and take an atlas $\{(U_\alpha, \phi_\alpha)\}$ for M. Let Y_α be the vector field on U_α defined by $Y_\alpha = d\phi_\alpha^{-1}(e_1)$, where e_1 is the unit vector along the first coordinate axis. Let $\{\lambda_a\}$ be a partition of unity subordinate to the covering $\{U_\alpha\}$. Then $X = \sum_\alpha \lambda_\alpha Y_\alpha$ is the desired vector field, and working with this X, we may get the following analogue of the above lemma for manifolds with boundary.

Lemma 3.4. *Let X be the vector field constructed above on a manifold with boundary M. Then there is a positive smooth function $\epsilon : M \longrightarrow \mathbb{R}$ and a smooth map $f : W_+(\epsilon) \longrightarrow M$, where $W_+(\epsilon) = \{(x,t) \in M \times \mathbb{R}_+ \mid t < \epsilon(x)\}$ satisfying the conditions* (1) *and* (2) *of Lemma 3.3.*

Proof of Theorem 3.2. Construct the vector field X and the map f of Lemma 3.4. Clearly, f maps $\partial M \times \{0\}$ diffeomorphically onto ∂M, and df is an isomorphism at each point of $\partial M \times \{0\}$, because X is inward pointing along ∂M. Therefore by Lemma 3.3, f is an embedding of a neighbourhood of $\partial M \times \{0\}$ into M. This neighbourhood contains a $W_+(\epsilon)$ for some positive smooth function ϵ. There is a diffeomorphism $W_+(\epsilon) \longrightarrow M \times (0,1)$ given by

$$(x,t) \mapsto \Big(x, \frac{t}{\epsilon(x)}\Big),$$

and a diffeomorphism $M \times [0,1) \longrightarrow M \times [0,\infty)$ given by

$$(x,t) \mapsto \Big(x, \frac{t}{1-t}\Big)$$

. This completes the proof of the collar neighbourhood theorem.

Remark 3.5. We may take a collar neighbourhood of ∂N as an embedding

$$\partial N \times [0,1) \longrightarrow N$$

(or even $\partial N \times [0,1] \longrightarrow N$) which is Id on ∂N.