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Lecture on Differential Topology Part 2

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1. Smooth partition of unity

Recall that a covering $\mathcal{U}$ of a topological space X is called **locally finite** if each point of X has an open neighbourhood which intersects only finitely many members of $\mathcal{U}$. Another covering $\mathcal{V}$ of X is called a **refinement** of $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$. A Hausdorff space X is **paracompact** if every open covering of X admits an open locally finite refinement.

Theorem 1.1. *Every manifold M is paracompact.*

Proof. Since M is locally homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+$, it is locally compact (each of its points has a compact neighbourhood). Then each point $x \in M$ has an open neighbourhood whose closure is compact. For, if U is an arbitrary open neighbourhood of x, and K a compact neighbourhood of x, then $V = U \cap \text{Int}K \subset K$, and so $\overline{V}$ is compact, being a closed subset of a compact set. It follows then, since M is second countable, that M admits a countable basis $\{V_j\}$ such that each $\overline{V}_j$ is compact.

Then, there is an increasing sequence $K_1 \subset K_2 \subset \cdots \subset K_j \subset \cdots$ of compact subsets whose union is M such that $K_j \subset \text{Int}K_{j+1}$, for each j. Indeed, we may take $K_1 = \overline{V}_1$, and, assuming inductively that K_j has been defined, if m is the smallest integer $> j$ such that $K_j \subset V_1 \cup \cdots \cup V_m$, then we may take

$$K_{j+1} = \overline{V}_1 \cup \cdots \cup \overline{V}_m = \overline{V_1 \cup \cdots \cup V_m}.$$

Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of M. Choose a locally finite refinement $\mathcal{V}$ as follows. Let $K_{-1} = K_0 = \emptyset$, and, for each $j \geq 0$, consider open sets

$$(\text{Int}K_{j+2} - K_{j-1}) \cap U_\alpha, \quad U_\alpha \in \mathcal{U}.$$

These open sets cover the compact set $K_{j+1} - \text{Int}K_j$. Therefore, we can find a finite subcover $\mathcal{V}_j = \{V_1^j, \ldots, V_{\alpha(j)}^j\}$ ($\alpha(j)$ an integer). The collection $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots$ covers M, since the sets $K_{j+1} - \text{Int}K_j$ cover M. The covering $\mathcal{V}$ is a refinement of $\mathcal{U}$. It is locally finite, because if $x \in K_j$, then $\text{Int}K_{j+1}$ is a neighbourhood of x which intersects no member of $\mathcal{V}_k$ for $k > j + 1$. □

Remark 1.2. Actually we have constructed a locally finite refinement $\mathcal{V}$ which is countable.

Lemma 1.3. *Any open covering $\{U_\alpha\}$ of a manifold M has a countable locally finite refinement by coordinate neighbourhoods, each of which has compact closure.*

Proof. The proof follows the same line of arguments as that of the above theorem. One has only to choose the covering of each compact set $K_{j+1} - \text{Int}K_j$ suitably. Each point x of the open set $(\text{Int}K_{j+2} - K_{j-1}) \cap U_\alpha$ has a coordinate neighbourhood V_x and a homeomorphism ϕ_x such that $V_x \subset (\text{Int}K_{j+2} - K_{j-1}) \cap U_\alpha$, and $\phi_x(V_x)$

contains a closed n-ball with centre at $\phi_x(x)$, $n = \dim M$. Let B_x^n be a smaller concentric closed n-ball, and $W_x = \phi_x^{-1}(\mathrm{Int}B_x^n)$. With this choice, $\overline{W}_x$ will be compact. We may find a finite number of W_x which cover $K_{j+1} - \mathrm{Int}K_j$, and then proceed as before. □

A covering $\{V_i\}$ is called a **shrinking** of a covering $\{U_i\}$ if each $\overline{V}_i \subset U_i$.

Lemma 1.4 (**Shrinking lemma).** *Let $\mathcal{U} = \{U_i\}_{i\geq 1}$ be a countable locally finite open covering of M. Then there is another open covering $\{V_i\}$ of M such that $\overline{V}_i \subset U_i$ for every $i \geq 1$.*

Proof. We may assume that M is connected. Now write $\mathcal{U}_k = \cup_{i\geq k} U_i$. and construct the open sets V_i inductively in the following way.

The closed set $A_1 = U_1 - \mathcal{U}_2$ is contained in U_1, and so $M = A_1 \cup \mathcal{U}_2$. Since M is normal, we may choose an open set V_1 such that $A_1 \subset V_1 \subset \overline{V}_1 \subset U_1$. Then $M = V_1 \cup \mathcal{U}_2$. Next, suppose that the open sets $V_1, \ldots, V_{k-1}$ have been chosen so that $\overline{V}_i \subset U_i$ for $i = 1, \ldots, k-1$, and $M = V_1 \cup \cdots \cup V_{k-1} \cup \mathcal{U}_k$. Then the closed set $A_k = U_k - (V_1 \cup \cdots \cup V_{k-1} \cup \mathcal{U}_{k+1})$ is contained in U_k, and we have

$$M = V_1 \cup \cdots \cup V_{k-1} \cup A_k \cup \mathcal{U}_{k+1}.$$

Choose an open set V_k such that $A_k \subset V_k \subset \overline{V}_k \subset U_k$. Then we have

$$M = V_1 \cup \cdots \cup V_k \cup \mathcal{U}_{k+1}.$$

To see that the collection $\{V_i\}$ is a covering of M, take any point $x \in M$. Then, since the covering $\mathcal{U}$ is locally finite, there is a largest m such that $x \notin U_k$ for $k > m$, that is, $x \notin \mathcal{U}_m$. Since $M = V_1 \cup \cdots \cup V_{m-1} \cup \mathcal{U}_m$, it follows that $x \in V_1 \cup \cdots \cup V_{m-1}$. This completes the proof. □

Theorem 1.5. *Every manifold is metrizable.*

Proof. Since every paracompact space is normal and a manifold is second countable, the proof follows trivially from Urysohn's metrization theorem, if the manifold is connected. This theorem states that every second countable regular space can be embedded topologically in infinite dimensional Hilbert coordinate space.

If a manifold M is not connected and has several components $\{M_\alpha\}$ with metric ρ_α on M_α making it a metric space, then a metric ρ in M is obtained by

$$\begin{aligned} \rho(x,y) &= \min(\rho_\alpha(x,y), 1) \text{ if } x,\ y \in M_\alpha \\ &= 1 \text{ if } x \in M_\alpha, y \in M_\beta \text{ and } \alpha \neq \beta. \end{aligned}$$

Clearly the metric ρ is compatible with the topology of M. □

Locally the topology of M is the same as the topology of $\mathbb{R}^n$, and therefore it is given by the standard metric in $\mathbb{R}^n$. If (U, ϕ) is a coordinate chart about a point $p \in M$ with coordinates $(x_1, \ldots, x_n)$ such that $\phi(U)$ is a convex set in $\mathbb{R}^n$, then we may write for $x \in U$

$$\rho(x,p) = [(x_1 - x_1(p))^2 + \cdots + (x_n - x_n(p))^2]^{\frac{1}{2}},$$

or equivalently,

$$\rho(x,p) = \max\{|x_i - x_i(p)|\}.$$

We shall have occasions to use a bump function whose definition is as follows.

Definition 1.6. A **bump function** is a smooth function $\mathcal{B} : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\mathcal{B}(x) = 0 \text{ if } x \leq 0,\ 0 < \mathcal{B}(x) < 1 \text{ if } 0 < x < 1,\ \mathcal{B}(x) = 1 \text{ if } x \geq 1.$$

To construct a bump function $\mathcal{B}$, first define

$$\begin{aligned} \psi(x) &= \exp\left(\frac{1}{x(x-1)}\right) \text{ if } 0 < x < 1 \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then ψ is smooth, non-negative, and non-vanishing when $0 < x < 1$. Now define $\mathcal{B}$ by

$$\mathcal{B}(x) = \frac{\int_0^x \psi(t)dt}{\int_0^1 \psi(t)dt}.$$

Definition 1.7. The **support** of a function $f : M \longrightarrow \mathbb{R}$, denoted by supp$f$, is the closure of the set of points of M where f is non-zero.

Lemma 1.8. *If $K \subset U \subset M$, where K is compact and U is open, then there is a smooth function $\mu : M \longrightarrow [0, \infty)$ such that $\mu(x) > 0$ if $x \in K$, and* supp $\mu \subset U$.

Proof. Define a smooth function $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ by $\alpha(x) = \mathcal{B}(1 - |x|)$. Then $\alpha(x) > 0$ if $|x| < 1$, and $\alpha(x) = 0$ if $|x| \geq 1$. Now construct for each $p \in K \subset U$ a smooth function $\mu_p : M \longrightarrow [0, \infty)$ such that $\mu_p(p) > 0$ and supp $\mu_p \subset U$ in the following way. Choose local coordinates $(x_1, \ldots, x_n)$ about p with p corresponding to the origin such that $B_r = \{(x_1, \ldots, x_n) \mid |x_i| < r\} \subset U$, for a suitable $r > 0$. Define μ_p by

$$\begin{aligned} \mu_p(x) &= \alpha\Big(\frac{x_1}{r}\Big) \cdots \alpha\Big(\frac{x_n}{r}\Big) \text{ if } x \in B_r \\ &= 0 \text{ otherwise.} \end{aligned}$$

As p runs over K the open sets $\{x \in M \mid \mu_p(x) > 0\}$ cover K. By compactness, a finite number of them still cover K. Then the sum of the corresponding finite number of functions μ_p is the required function μ. □

Definition 1.9. Let M be a manifold with an open covering $\mathcal{U} = \{U_i\}_{i \in A}$. Then a **smooth partition of unity** subordinate to $\mathcal{U}$ is a family of smooth functions $\{\lambda_i : M \longrightarrow \mathbb{R}\}$ satisfying the following conditions.

(i) supp$\lambda_i \subset U_i$ for all $i \in A$,
(ii) $0 \leq \lambda_i(x) \leq 1$ for all $x \in M$ and $i \in A$,
(iii) each $x \in M$ has a neighbourhood on which all but finitely many functions λ_i are identically zero,
(iv) $\sum_{i \in A} \lambda_i(x) = 1$ for all $x \in M$ (note that the sum is always finite by (iii)).

Lemma 1.10. *Let $\mathcal{U} = \{U_i\}_{i \in A}$ and $\mathcal{V} = \{V_j\}_{j \in B}$ be two open coverings of M such that $\mathcal{U}$ refines $\mathcal{V}$. Then, if $\mathcal{U}$ has a subordinate smooth partition of unity, so has $\mathcal{V}$.*

Proof. Let $\{\lambda_i\}_{i \in A}$ be a partition of unity subordinate to $\mathcal{U}$. Let $f : A \longrightarrow B$ be a map of the index sets so that $U_i \subset V_{f(i)}$. Define $\mu_j : M \longrightarrow \mathbb{R}$ by

$$\mu_j(x) = \sum_{i \in f^{-1}(j)} \lambda_i(x).$$

It ia easily checked that the conditions (i) - (iv) hold for the family $\{\mu_j\}$, when U_i are replaced by V_j. □

Remark 1.11. Some people call $\{\lambda_i\}$ a partition of unity subordinate to $\mathcal{V}$, In this case the condition (i) has to be replaced by the following condition:

"for every $i \in A$ there is a $j \in B$ such that supp $\lambda_i \subset V_j$".

Theorem 1.12. *Any manifold M with an open covering $\{U_i\}$ admits a smooth partition of unity subordinate to $\{U_i\}$.*

Proof. We may assume that the given covering $\{U_i\}$ is countable and locally finite such that each of its members U_i is a coordinate neighbourhood with compact closure (Lemma 1.3). We may find another open covering $\{V_i\}$ of M such that $\overline{V}_i \subset U_i$ (Lemma 1.4). Now construct smooth functions $\mu_i : M \longrightarrow \mathbb{R}$ as described in Lemma 1.8 such that $\mu_i > 0$ on $\overline{V}_i$ and supp $\mu_i \subset U_i$. Then the function $\sum_i \mu_i$ is a well-defined positive smooth function, and the family of functions $\lambda_i = \mu_i / \sum_i \mu_i$ is the required smooth partition of unity. □

Lemma 1.13. *If $\{\lambda_i : U_i \longrightarrow \mathbb{R}\}$ is a smooth partition of unity on M, and $\{f_i : U_i \longrightarrow \mathbb{R}\}$ is a family of smooth functions, then the function $f : M \longrightarrow \mathbb{R}$ defined by $f(x) = \sum_i \lambda_i(x) f_i(x)$ is smooth.*

Proof. Since the function $\lambda_i f_i$ is smooth on U_i and vanishes on a neighbourhood of $M - U_i$, it can be extended over M using the zero function on $M - U_i$. Therefore the sum $f = \sum_i \lambda_i f_i$ is smooth. □

Lemma 1.14. *For a map $f : U \longrightarrow \mathbb{R}^m$, where U is open in $\mathbb{R}^n_+$, the following conditions are equivalent.*

(*a*) *f is smooth, as defined in Definition 2.4 (Part 1) using local extendability condition,*

(*b*) *there is an open set V in $\mathbb{R}^n$ and a smooth map $F : V \longrightarrow \mathbb{R}^m$ such that $V \cap \mathbb{R}^n_+ = U$ and $F|U = f$.*

Proof. The part (b) $\Rightarrow$ (a) is trivial. Next, assume (a). Then, for each $x \in U$, there is an open neighbourhood V_x of x in $\mathbb{R}^n$ and a smooth map $F_x : V_x \longrightarrow \mathbb{R}^m$ such that $f = F_x$ on $U \cap V_x$. Let $W = \cup_{x \in U} V_x$. Then W is open in $\mathbb{R}^n$, and $U \subset W$.

The manifold W admits a partition of unity $\{\lambda_x\}$ subordinate to the covering $\{V_x\}$. Then $G = \sum_{x \in U} \lambda_x F_x$ is a smooth map from W to $\mathbb{R}^m$. On the other hand, there exists an open set V' of $\mathbb{R}^n$ such that $U = V' \cap \mathbb{R}^n_+$. Then taking $V = W \cap V'$ and $F = G|V$, we get the condition (b). □

Lemma 1.15. *If $f : M \longrightarrow \mathbb{R}$ is a positive continuous function, then there is a smooth function $g : M \longrightarrow \mathbb{R}$ such that*

$$0 < g(x) < f(x) \text{ for all } x \in M.$$

When M is compact, g may be taken to be a constant function.

Proof. As in the proof of Theorem 1.12, consider a locally finite covering $\{U_i\}$ of M, and another open covering $\{V_i\}$ such that $\overline{V}_i$ is compact and $\overline{V}_i \subset U_i$. Take a smooth partition of unity $\{\lambda_i\}$ such that $\lambda_i > 0$ on $\overline{V}_i$ and supp $\lambda_i \subset U_i$. Choose $\delta_i > 0$ smaller than the infimum of f on the compact set $\overline{V}_i$, and define $g : M \longrightarrow \mathbb{R}$ by $g(x) = \sum_i \delta_i \lambda_i(x)$. Then g is smooth. Since the sum $\sum_i \lambda_i(x)$ is finite and equal to 1, and the maximum of the corresponding δ_i is less than $f(x)$, we have $g(x) < f(x)$. Also $g(x) > 0$, since all δ_i are so. □

Lemma 1.16. *If K is a closed subset of a manifold M, and $f : K \longrightarrow \mathbb{R}$ is a smooth function, then f extends to a smooth function $F : M \longrightarrow \mathbb{R}$.*

Proof. In view of Definition 2.4 (Part 1), cover K by open sets U_i such that there exist smooth functions $g_i : U_i \longrightarrow \mathbb{R}$ with $g_i = f$ on $K \cap U_i$. The sets U_i and $M - K$ form an open covering of M. Let $\{\lambda_i\}$ be a smooth partition of unity subordinate to this covering. Then the smooth extension F of f is given by

$$F(x) = \begin{cases} \sum_i \lambda_i(x) g_i(x), & \text{if } x \notin M - K \\ 0, & \text{otherwise} \end{cases}$$

□

Lemma 1.17 (Smooth Urysohn's lemma). *If $K \subset U \subset M$, K closed, U open, then there is a smooth function $f : M \longrightarrow \mathbb{R}$ such that $0 \le f \le 1$, $f|K = 1$ and supp $f \subset U$.*

Proof. The open sets $U_1 = U$ and $U_2 = M - K$ form a covering of M. Let $\lambda_1 : U_1 \longrightarrow \mathbb{R}$ and $\lambda_2 : U_2 \longrightarrow \mathbb{R}$ be a smooth partition of unity subordinate to this covering. Then λ_1 extended over M by the zero function outside U_1 is a solution f of the problem. □

Theorem 1.18 (Whitney's Weak Embedding Theorem). . *If M is a compact n-manifold, then there is an embedding $f : M \longrightarrow \mathbb{R}^m$, where $m = r(n+1)$ for some integer $r > 0$.*

Proof. Find a finite covering of M by coordinate charts (U_i, ϕ_i), $i = 1, \dots, r$, and open sets V_i also covering M such that $\overline{V_i} \subset U_i$ for all i. By Lemma 1.17, there are C^∞ functions $\lambda_i : M \longrightarrow \mathbb{R}$ such that $\lambda_i|\overline{V_i} = 1$ and supp $\lambda_i \subset U_i$. Let $\psi_i : M \longrightarrow \mathbb{R}^n$ be C^∞ maps given by

$$\begin{aligned} \psi_i(p) &= \lambda_i(p)\phi_i(p) \text{ if } p \in U_i \\ &= 0 \text{ otherwise.} \end{aligned}$$

Define $f : M \longrightarrow \mathbb{R}^{r(n+1)}$ by

$$f(p) = (\psi_1(p), \dots, \psi_r(p), \lambda_1(p), \dots, \lambda_r(p)), p \in M.$$

Then the Jacobian matrix $J(f)$ has rank n at every point $p \in M$. Because, $J(\phi_i)$ has rank n, and if $p \in M$, then

$$(p \in V_i \text{ for some } i) \Rightarrow (\lambda_i(p) = 1) \Rightarrow (\phi_i = \psi_i) \Rightarrow (J(\phi_i) = J(\psi_i) \text{ at } p),$$

so $J(\psi_i)$, and hence $J(f)$, has rank n at p.

Also f is injective. Because, if $f(p) = f(q)$ then $\psi_i(p) = \psi_i(q)$ and $\lambda_i(p) = \lambda_i(q)$ for all i, and if $p \in V_j$ then $\lambda_j(p) = \lambda_j(q) = 1$, and so $\phi_j(p) = \phi_j(q)$, which implies $p = q$, ϕ_j being injective.

Thus f is an injective immersion. Since M is compact and $f(M)$ is Hausdorff, f is a homeomorphism onto its image, and hence it is an embedding. □

We remark that the theorem is unsatisfactory in that the value of m, which is the dimension of the Euclidean space, depends on the number r of coordinate neighbourhoods required to cover M, and therefore m may be much larger than we would like it. A stronger version of this theorem says that any n-manifold can be embedded in $\mathbb{R}^{2n+1}$ as a subspace.

Proposition 1.19. *Any metric on a manifold M compatible with the topology of M can be turned into a complete metric giving the same topology of M.*

Proof. The proof will be clear from the next two lemmas. □

Recall that a continuous map between topological spaces is **proper** if the inverse image of any compact set is compact.

Lemma 1.20. *If X is a metric space with metric ρ and $f : X \longrightarrow \mathbb{R}$ is a continuous proper map, then the map $\rho' : X \times X \longrightarrow \mathbb{R}$ given by*

$$\rho'(x, y) = \rho(x, y) + |f(x) - f(y)|, x, y \in X$$

is a complete metric on X which is compatible with the original topology of X.

Proof. Clearly ρ' is a metric. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies on X induced by the metrics ρ and ρ' respectively. Then, since ρ' is continuous with respect to $\mathcal{T}$, we have $\mathcal{T}' \subset \mathcal{T}$. Conversely, take an open set U in $\mathcal{T}$ and a point x in U. Then we can find an $\epsilon > 0$ so that

$$B'(x, \epsilon) = \{y \in X \mid \rho'(x, y) < \epsilon\} \subset B(x, \epsilon) = \{y \in X \mid \rho(x, y) < \epsilon\} \subset U.$$

This means U is in $\mathcal{T}'$, and $\mathcal{T} = \mathcal{T}'$. Next, to see that ρ' is complete, take a Cauchy sequence $\{x_n\}$ in X with respect to the metric ρ'. Then there is a number $m > 0$ such that $\rho'(x_1, x_n) < m$ for all $n \geq 1$. Therefore $|f(x_1) - f(x_n)| < m$ for all $n \geq 1$, and so

$$x_n \in f^{-1}([f(x_1) - m, f(x_1) + m]).$$

Since f is proper, the above set is compact, and the sequence $\{x_n\}$ converges to a limit in X. □

Lemma 1.21. *On a manifold M there always exists a proper smooth function $f : M \longrightarrow \mathbb{R}$.*

Proof. Find an open covering of M by open sets with compact closure, and a smooth partition of unity $\{\lambda_i\}$ subordinate to a locally finite refinement of this covering. Since the refinement is countable, we may assume that the functions λ_i are indexed by integers $i > 0$. Then the function $f : M \longrightarrow [1, \infty)$ given by $f(x) = \sum_i i\lambda_i(x)$ is a well-defined, because all but a finite number of $\lambda_i(x)$ vanish. Now $f(x) \leq k$ implies at least one of the k functions $\lambda_1, \ldots, \lambda_k$ must not vanish at x (if all of them were zero, then $f(x)$ would be $\geq k + 1$). Therefore $f^{-1}([-k, k])$ is contained in the set $\cup_{i=1}^{k}\{x \in M : \lambda_i(x) \neq 0\}$ which has compact closure. This implies that f is proper, because every compact subset of $\mathbb{R}$ is contained in some interval $[-k, k]$. □

We now describe some more facts about proper maps.

Recall that if X is a locally compact Hausdorff space, then its one-point compactification is a space $X^+ = X \cup \{\infty\}$ (∞ represents a point not in X) such that the topology of X^+ comprises all open sets in $X \subset X^+$ and all sets of the form $X^+ - K$, where K is a compact subset of X. This is the unique topology in X^+ which makes it a compact Hausdorff space with X as a subspace. If X is already compact, then ∞ is an isolated point in X^+.

Lemma 1.22. *Let $f : X \longrightarrow Y$ be a continuous map between locally compact Hausdorff spaces, and $f^+ : X^+ \longrightarrow Y^+$ be its extension obtained by setting $f^+(\infty) = \infty$. Then f is proper if and only if f^+ is continuous.*

Proof. Suppose that a continuous map f is proper, and U is an open set in Y^+. Then, if $U \subset Y$, $(f^+)^{-1}(U) = f^{-1}(U)$ is open, and if $U = Y^+ - K$ with $K \subset Y$ compact, then $(f^+)^{-1}(U) = X^+ - f^{-1}(K)$ is also open, since $f^{-1}(K)$ is compact, and hence closed. Therefore f^+ is continuous.

Conversely, suppose f^+ is continuous, and $K \subset Y$ is a compact set. Then K is compact, and hence closed in Y^+. Then $(f^+)^{-1}(K)$ is closed in X^+, and hence compact and contained in X. Thus f is proper. □

Corollary 1.23. *Let X be locally compact Hausdorff, and Y Hausdorff. Then a continuous injective proper map $f : X \longrightarrow Y$ is a homeomorphism onto its image.*

Proof. Since f is a continuous proper map from X onto $Z = f(X)$, it extends to a continuous map of the one-point compactifications $f^+ : X^+ \longrightarrow Z^+$. Since f^+ is a bijection from a compact space onto a Hausdorff space, it is a homeomorphism. Therefore f is a homeomorphism onto its image. □

Corollary 1.24. *A proper injective immersion from a manifold M into a manifold N is an embedding.*

Proof. An injective immersion which is a homeomorphism onto its image is an embedding. □

Corollary 1.25. *Any continuous proper map $f : X \longrightarrow Y$ between locally compact Hausdorff spaces is a closed map.*

Proof. Let C be any closed set in X, and $D = f(C)$. Then $C^+ = C \cup \{\infty\}$ is closed, and hence compact, in X^+. Since f^+ is continuous, $f^+(C^+)$ is compact, and hence closed, in Y^+. Therefore $f(C) = f^+(C^+) \cap Y$ is closed in Y. □

2. Riemannian Metric

Definition 2.1. A **Riemannian metric** g on a manifold M is a smooth positive definite symmetric 2-tensor field on M. This assigns to each point $p \in M$ a positive definite symmetric bilinear form or inner product on the tangent space $\tau(M)_p$

$$g_p : \tau(M)_p \times \tau(M)_p \longrightarrow \mathbb{R}.$$

Recall that positive definiteness means $g_p(v, v) > 0$ for all non-zero $v \in \tau(M)_p$, and $g_p(u, u) = 0$. A **Riemannian manifold** is a manifold with a Riemannian metric on it.

The length of a tangent vector $v \in \tau(M)_p$ is then defined in the usual way as

$$\|v\| = g_p(v,v)^{1/2}.$$

In terms of local coordinate system $(x_1, \ldots, x_n)$ in M with basic vector fields $\delta_i = \partial/\partial x_i$, the local representation of g is given by

$$g = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j,$$

where $g_{ij} = g(\delta_i, \delta_j)$ are real-valued functions on the coordinate neighbourhood U of the system. If $v, w \in \tau(M)_p$, and $v = \sum_i v_i \delta_i, w = \sum_j w_j \delta_j$, then $g_p(v,w) = \sum_{i,j} g_{ij}(p) v_i w_j$. As in the case of 2-forms, g is smooth if and only if for every pair of vector fields X, Y on U, the function $g(X,Y)$ is smooth on U. Also g is $\mathbb{R}$-bilinear, as well as $C^\infty(M)$-bilinear:

$$g(fX, Y) = fg(X,Y) = g(X, fY), \quad \text{for } f \in C^\infty(M).$$

Definition 2.1 is equivalent to the following. A Riemannian metric g on M is a positive definite symmetric $C^\infty(M)$-bilinear map

$$g : \ni\!\!\in(M) \times \ni\!\!\in(M) \longrightarrow C^\infty(M),$$

where $\ni\!\!\in(M)$ denotes the algebra (over the ring $C^\infty(M)$) of all smooth vector fields on M.

Example 2.2. The Euclidean space $\mathbb{R}^n$ with coordinates

$$(u_1, \ldots, u_n)$$

has a natural Riemannian metric

$$g = \sum_{i,j=1}^{n} \delta_{ij} du_i du_j = \sum_{i=1}^{n} (du_i)^2.$$

Theorem 2.3. *Any manifold M admits a Riemannian metric.*

Proof. Choose an open covering of M by coordinate neighbourhoods $\{U_i\}$. Let $x_{i1}, \ldots, x_{in}$ be local coordinates in U_i. Using these coordinates define a metric g_i on U_i by

$$g_i = (dx_{i1})^2 + \cdots + (dx_{in})^2.$$

Let $\{\lambda_i\}$ be a smooth partition of unity subordinate to the covering $\{U_i\}$. Then

$$g = \sum_i \lambda_i g_i$$

is a well defined Riemannian metric on M. □

The necessity of the condition of paracompactness on M in the above theorem may be seen from the following negative result. The Alexandroff line or the long line is a smooth connected manifold of dimension one, but not a manifold in our sense (it is not paracompact). It is known that this manifold cannot be given a Riemannian structure (see H. Kneser, Analytische Struktur und Abzählbarkeit, Ann. Acad. Sci. Fenn, Ser. A. I. 251/5 (1958)).

The importance of Riemannian metric is that it turns the tangent space at each point into an inner product space, which enables us to define angle between curves (that is, angle between tangent vectors of the curves at the point of intersection), and length of curves.

We shall use the terms 'path', 'curve', and 'parametrized curve' synonymously. Note that a curve $\sigma : [a,b] \longrightarrow M$ on a closed interval $[a,b]$ is smooth if it can be extended to a smooth map on an open interval containing $[a,b]$. If $\sigma : [a,b] \longrightarrow M$ is a smooth curve in a Riemannian manifold M, then its **length** $\ell(\sigma)$ is defined by

$$\ell(\sigma) = \int_a^b \|\dot\sigma(t)\| dt = \int_a^b \sqrt{g(\dot\sigma(t), \dot\sigma(t))} dt, \quad \dot\sigma = \frac{d\sigma}{dt}.$$

A **reparametrization** of σ is a curve $\sigma\circ\phi : [c,d] \longrightarrow M$, where $\phi : [c,d] \longrightarrow [a,b]$ is a diffeomorphism with positive derivative everywhere (i.e. an orientation preserving diffeomorphism[1]). Then $\sigma(t)$, $t \in [a,b]$, and $\sigma(\phi(u))$, $u \in [c,d]$ trace the same curve in M in the same direction. Also $\ell(\sigma) = \ell(\sigma\circ\phi)$, by the change of variable formula for integrals: if $t = \phi(u)$, $u \in [c,d]$ new parameter, then $d\sigma/du = (d\sigma/dt)\cdot(dt/du)$, and

$$\begin{aligned}\int_c^d \left[g\Big(\frac{d\sigma}{du}, \frac{d\sigma}{du}\Big)\right]^{\frac12} du &= \int_a^b \left[g\Big(\frac{d\sigma}{dt}, \frac{d\sigma}{dt}\Big)\cdot\Big(\frac{dt}{du}\Big)^2\right]^{\frac12} \frac{du}{dt}\, dt \\ &= \int_a^b \left[g\Big(\frac{d\sigma}{dt}, \frac{d\sigma}{dt}\Big)\right]^{\frac12} dt.\end{aligned}$$

Thus the length of a curve is an invariant under reparametrization. Any curve σ can be reparamtrized with its arc length s as parameter, where

$$s = \phi(t) = \int_a^t \|\dot\sigma(t)\| dt$$

is a strictly increasing function $[a,b] \longrightarrow [0, \ell(\sigma)]$. The reparametrization $\sigma\circ\phi^{-1} : [0,\ell(\sigma)] \longrightarrow M$ has tangent vectors of unit length at all points, as may be seen by the chain rule, and therefore $\ell(\sigma\circ\phi^{-1} \mid [0,s]) = s$ for all $s \in [0, \ell(\sigma)]$.

A continuous map $\sigma : [a,b] \longrightarrow M$ is called a **piecewise smooth curve** if there is a partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a,b]$ such that $\sigma \mid [t_{i-1}, t_i]$ is smooth for $i = 1,\ldots,k$ (the left- and right-hand derivatives of σ at $t_1,\ldots,t_{n-1}$ may be different). Then the length $\ell(\sigma)$ is given by $\ell(\sigma) = \sum_i \ell(\sigma \mid [t_{i-1}, t_i])$, that is, by $\int_a^b \|\dot\sigma(t)\| dt$ as in the smooth case.

Exercise 2.4. For a piecewise smooth curve $\sigma : [a,b] \longrightarrow M$, define a continuous non-decreasing function $\phi : [a,b] \longrightarrow \mathbb{R}$ by

$$\phi(u) = \int_a^u \|\dot\sigma(t)\| dt.$$

(a) Show that ϕ is smooth at every point u where $\dot\sigma(u)$ exists and is non-zero.

(b) Show that if ℓ denotes the length of σ, then the map $\sigma\circ\phi^{-1} : [0,\ell] \longrightarrow M$ is well-defined and continuous, even if ϕ^{-1} may not be a function. Moreover, $\sigma\circ\phi^{-1}$ is smooth at a point $\phi(u)$ such that $\dot\sigma(u) \neq 0$.

The next lemma concerns connectedness of manifolds. First note that for a manifold pathwise connectedness and connectedness are the same. A pathwise connected space is necessarily connected, because any two points lie in a connected

[1]The concept of orientation is discussed in §2, Part 3.

subset of the space (continuous image of an interval being connected). Conversely, in a locally Euclidean space, connectedness implies pathwise connectedness, by an argument that we shall describe in the next lemma.

Two points p and q in M are called piecewise smoothly connected if there is a piecewise smooth curve in M whose image contains p and q.

Lemma 2.5. *For a manifold M 'joining by continuous curve' is equivalent to 'joining by piecewise smooth curve'.*

Proof. Clearly joining by piecewise smooth curve is an equivalence relation. An equivalence class is an open set in M, because if (U, ϕ) is a coordinate chart about a point $p \in M$ with $\phi(U) = V$ a convex set in an Euclidean space, then any point of U can be joined to p by a curve corresponding to a straight line in V. Again, an equivalence class is a closed set in M, because it is the complement of the union of other equivalence classes. It follows that a subset of M is open and closed if and only if it is a union of equivalence classes. This completes the proof. □

Remark 2.6. The lemma also holds if 'joining by piecewise smooth curve' is replaced by 'joining by smooth curve'. Because 'joining by smooth curve' is also an equivalence relation. This may be seen in the following way.

The relation is obviously reflexive and symmetric. To see transitivity, suppose that σ and τ are two smooth curves with images containing the pairs of points (x, y) and (y, z) respectively. Suppose without loss of generality that

$$\sigma(-1) = x,\ \sigma(0) = y = \tau(0),\ \tau(1) = z.$$

Let (U, ϕ) be a coordinate chart about y such that $\phi(U) = V$ is convex. Since σ and τ are continuous, there exists an ϵ with $0 < \epsilon < 1$ such that $|t| < \epsilon$ implies $\sigma(t),\ \tau(t) \in U$. Let $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function which is 0 near $t = -\epsilon$ and 1 near $t = \epsilon$. For example, we may take

$$\lambda(t) = \mathcal{B}\left(\frac{t}{\epsilon} + \frac{1}{2}\right),$$

where $\mathcal{B}$ is a bump function (Definition 1.6). Define a curve ω by

$$\begin{aligned} \omega(t) &= \sigma(t) \text{ if } t < -\epsilon \\ &= (1 - \lambda(t))\sigma(t) + \lambda(t)\tau(t) \text{ if } -\epsilon < t < \epsilon \\ &= \tau(t) \text{ if } t > \epsilon \end{aligned}$$

where the second line is a convex combination in V. Then ω is smooth and its image contains x and y.

The equivalence classes are components of M, and M is connected if it has only one component.

If M is connected, a Riemannian metric g on M induces a metric

$$d : M \times M \longrightarrow \mathbb{R}$$

so that (M, d) becomes a metric space. The metric d is defined in the following way. Since M is connected, every pair of points $p_1, p_2 \in M$ can be joined by a piecewise smooth curve σ. Then $d(p_1, p_2)$ is defined by

$$d(p_1, p_2) = \inf \ell(\sigma),$$

where the infimum is taken over all piecewise smooth curves σ from p_1 to p_2. Clearly d is a pseudo-metric, that is, it is symmetric, and satisfies the triangle inequality and the condition that $d(p_1, p_2) = 0$ whenever $p_1 = p_2$. The triangle inequality follows, because if σ_1 and σ_2 are any two piecewise smooth curves from p_1 to p_2 and p_2 to p_3 respectively, then $d(p_1, p_3) \leq \ell(\sigma_1) + \ell(\sigma_2)$.

Theorem 2.7. *Let g be a Riemannian metric on a connected manifold M, and d be the induced pseudo-metric on M. Then d is a metric.*

We first prove an easy lemma.

Lemma 2.8. *Let σ be a piecewise smooth curve in an Euclidean space $\mathbb{R}^n$ from the origin to a point on the sphere of radius r centred at the origin. Then $\ell(\sigma) \geq r$, where $\ell(\sigma)$ is the length of σ with respect to the standard metric on $\mathbb{R}^n$. The equality holds if σ is a straight line segment.*

Proof. We have $\ell(\sigma) = \int_0^1 \|\dot{\sigma}(t)\| dt \geq \| \int_0^1 \dot{\sigma}(t) dt\| = \|\sigma(1) - \sigma(0)\| = r$. □

Proof of the theorem. We have only to show that $d(p_1, p_2) = 0$ implies that $p_1 = p_2$. Suppose that $p_1 \neq p_2$. Let $(x_1, \dots, x_n)$ be a coordinate system in a coordinate neighbourhood U about p_1. Let B be an open ball with respect to these coordinates with centre at p_1 such that $\overline{B} \subset U$ (B is obtained in the following way: if the coordinate chart is (U, ϕ) with $\phi(p_1) = 0$, the origin in $\mathbb{R}^n$, and B_0 is an open ball in $\mathbb{R}^n$ with centre at 0 such that $\overline{B_0} \subset \phi(U)$, then take $B = \phi^{-1}(B_0)$). Suppose that B is such that $p_2 \notin B$. Define a function $f : \tau(U) = U \times \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$f(p, v_1, \dots, v_n) = \Big[\sum_{i,j} g_{ij}(p) v_i v_j \Big]^{\frac{1}{2}} = \Big\| \sum_i v_i \frac{\partial}{\partial x_i}(p) \Big\|,$$

where $\| \ \|$ is the norm defined by the Riemanian metric g on U. Then $f|U \times S^{n-1}$ is positive and continuous, and therefore we can find a $k > 0$ such that

$$\frac{1}{k} \leq f|\overline{B} \times S^{n-1} \leq k.$$

Let $\| \ \|'$ be the Euclidean norm in $U \times \mathbb{R}^n$. Then since

$$\Big\| \sum_i v_i \frac{\partial}{\partial x_i}(p) \Big\|' = \Big[\sum_i v_i^2 \Big]^{\frac{1}{2}}$$

is 1 on $\overline{B} \times S^{n-1}$, we have

$$\frac{1}{k} \Big\| \sum_i v_i \frac{\partial}{\partial x_i}(p) \Big\|' \leq \Big\| \sum_i v_i \frac{\partial}{\partial x_i}(p) \Big\| \leq k \Big\| \sum_i v_i \frac{\partial}{\partial xi}(p) \Big\|' \tag{2.1}$$

on $\overline{B} \times S^{n-1}$, and hence on $\overline{B} \times \mathbb{R}^n$ since we may replace v_i by λv_i, $\lambda \in \mathbb{R}$, in these inequalities (the expressions in (1) being homogeneous in the v_i's).

Now, let σ be a piecewise smooth curve from p_1 to p_2, and σ_1 be the portion of σ within $\overline{B}$ from p_1 to the first point of intersection of σ and the boundary of B. Let r be the radius of B.

We have then

$$d(p_1, p_2) = \inf \|\sigma\| \geq \inf \|\sigma_1\| \geq \frac{1}{k} \inf \|\sigma_1\|' \geq \frac{1}{k} r > 0,$$

by the first inequality of (1) and the above lemma. Thus $p_1 \neq p_2 \Rightarrow d(p_1, p_2) \neq 0$. This shows that d is a metric on M. □

Theorem 2.9. *The topology on a connected Riemannian manifold M defined by the metric d is equivalent to the topology of M as a manifold. Therefore d is a continuous function on $M \times M$.*

Proof. The manifold topology has a basis consisting of coordinate neighbourhoods. Let p be an arbitrary point of M, and (U, ϕ) a coordinate chart at p. Then any open neighbourhood of p in U has two topologies, namely, the metric topology induced by the metric d, and the manifold topology induced by the Euclidean metric d' via the homeomorphism ϕ. It is enough to show that these topologies are the same, or explicitly, to show that there is an open neighbourhood B of p with $\overline{B} \subset U$ and a number $k > 0$ such that

$$\frac{1}{k} d' \leq d \leq k\, d', \text{ on } B \times B. \tag{2.2}$$

This will complete the proof of the theorem.

Let B be an open ball with respect to the coordinates in U with centre at p such that $\overline{B} \subset U$. Then, as shown in (1) of thr previous theorem, there is a number $k > 0$ such that

$$\frac{1}{k}\|v\|' \leq \|v\| \leq k\|v\|'$$

for any vector $v \in \tau(M)_q$ with $q \in \overline{B}$, where $\|v\|'$ and $\|v\|$ are the norms with respect to the metrics d' and d respectively. This means that for any piecewise smooth curve σ lying entirely in B, we have

$$\frac{1}{k}\|\sigma\|' \leq \|\sigma\| \leq k\|\sigma\|'.$$

Therefore we need only to be concerned about curves that do not lie entirely in B. We shall show by reducing the size of B that for any piecewise smooth curve τ having end points in B but goes outside of B, there is a piecewise smooth curve σ with the same end points lying entirely in B such that $\|\tau\| \geq \|\sigma\|$. This will enable us to compute d on $B \times B$ by restricting ourselves only to curves lying entirely in B. This will establish (2), and complete the proof of the theorem.

For this purpose, suppose that r is the radius of the ball B, and r_1 is a number such thst $r_1 \leq r/(2k^2 + 1)$. Let B_1 be a concentric open ball of radius r_1 such that $\overline{B_1} \subset B$. Let p_1 and p_2 be points in B_1, and τ be a piecewise smooth curve in U from p_1 to p_2. Let τ_1 be the portion of τ in B from p_1 to the first point of intersection of τ with the boundary of B, and σ be the straight line segment from p_1 to p_2 in B_1.

We have then

$$\|\tau\| \geq \|\tau_1\| \geq \frac{1}{k}\|\tau_1\|' \geq \frac{1}{k}(r - r_1) \geq 2kr_1 \geq k\|\sigma\|' \geq \|\sigma\|,$$

and we get what we wanted to show. □

3. Smooth approximations to continuous maps

Definition 3.1. Let M be a manifold, and N be a manifold with a metric ρ. Let $\delta : M \longrightarrow \mathbb{R}$ be a positive continuous function, and f, g be smooth maps from M to N. Then g is called a **δ-approximation** to f if

$$\rho(f(x), g(x)) < \delta(x),$$

for all $x \in M$.

The (fine) **C^0-topology** on the set $C^\infty(M, N)$ of smooth maps from M to N is a topology where the neighbourhood basis of $f \in C^\infty(M, N)$ comprises all sets of the form

$$B_0(f, \delta) = \{g \in C^\infty(M, N) \mid \rho(f(x), g(x)) < \delta(x)\}.$$

Thus g is a δ-approximation to f, if $g \in B_0(f, \delta)$.

The C^0 topology can be extended to the superset $C^0(M, N)$ of all continuous maps from M to N. It can be shown that this topology on $C^0(M, N)$ is larger than the compact open topology on $C^0(M, N)$, and that the C^0 topology on $C^\infty(M, N)$ does not depend on the choice of the metric on N (see AM,Topics in Differential Topology, §8.2, p. 262).

Lemma 3.2. *Let U be an open subset in $\mathbb{R}^n$ (or $\mathbb{R}^n_+$), and $f : U \longrightarrow \mathbb{R}$ be a continuous function such that f is smooth on an open set $V \subset U$. Let U' and V' be two other open sets in U such that $\overline{U}' \subset V'$, $\overline{V}' \subset U$, and $\overline{V}'$ is compact. Let $\delta : U \longrightarrow \mathbb{R}$ be a positive continuous function. Then there is a continuous function $g : U \longrightarrow \mathbb{R}$ such that g is smooth on $V \cup U'$, $g = f$ on $U - V'$, and $|g(x) - f(x)| < \delta(x)$ for all $x \in U$.*

The last condition means that g is a δ-approximation to f.

Proof. Let δ_0 be the positive minimum of the function δ on the compact set $\overline{V}'$. Then, by Weierstrass approximation theorem (see Dieudonné, Foundations of Modern Analysis, Academic Press, p. 139), there is a polynomial $p(x)$ such that

$$|p(x) - f(x)| < \delta_0 \text{ for } x \in \overline{V}'.$$

Let $h : U \longrightarrow \mathbb{R}$ be a smooth function such that $0 \leq h \leq 1$, $h = 1$ on $\overline{U}'$, and $h = 0$ on $U - V'$, as given by Lemma 1.17. Define $g : U \longrightarrow \mathbb{R}$ by

$$g(x) = h(x)p(x) + (1 - h(x))f(x), \ x \in U.$$

Then $g = f$ on $U - V'$, and $g = p$ on $\overline{U}'$. The last condition shows that g is smooth on U'. Also g is smooth on V, since f is smooth on it. Finally, on $\overline{V}'$ we have

$$|g(x) - f(x)| = |h(x)| \, |p(x) - f(x)| \ < \ \delta_0.$$

This completes the proof. □

Theorem 3.3 (**Smoothing Theorem**)**.** *Let M and N be manifolds. Let K be a smooth submanifold and a closed subset of M, and $f : M \longrightarrow N$ be a continuous map which is smooth on K. Then, there exist a positive continuous function $\delta : M \longrightarrow \mathbb{R}$, and a smooth map $g : M \longrightarrow N$ which agrees with f on K such that g is a δ-approximation to f.*

Remark 3.4. The possibility that $K = \emptyset$ is not ruled out.

Proof. For each $x \in M$, let A_x be a coordinate neighbourhood of x in M, and B_x be a coordinate neighbourhood of $f(x)$ in N such that $f(A_x) \subset B_x$. Let $C_x \subset A_x$ be the compact closure of a neighbourhood of x. We shall show that it is possible to choose a countable collection of such C such that their interiors cover M, and such that any C intersects only a finite number of the other C's of the collection.

For this purpose, we construct, as in the proof of Theorem 1.1, a sequence of compact sets $\{K_j\}$ covering M such that $K_j \subset \mathrm{Int}K_{j+1}$. Then the compact sets $L_j = K_j - \mathrm{Int}K_{j-1}$ also cover M, and $L_j \cap L_m = \emptyset$ if $m \neq j-1, j$, or $j+1$. For each $x \in L_j$, we choose coordinate neighbourhoods A_x, B_x, and a compact neighbourhood $C_x \subset A_x$ as above. By shrinking C_x, if necessary, we may suppose that it does not intersect L_m for $m \neq j-1, j$, or $j+1$. Choose a finite number of such C's whose interiors cover L_j, and doing this for each j construct a sequence of sets $\{C_n\}$ such that the $\mathrm{Int}C_n$ cover M and any member of the sequence intersects only a finite number of other members. Let $\{A_n\}$ and $\{B_n\}$ be the corresponding sequences of A_x's and B_x's respectively.

Define a sequence of closed sets S_k inductively as follows. Take S_0 as the given closed set K, and then take $S_k = S_{k-1} \cup C_k$, for $k \geq 1$. Then M is the union of the interiors of the sets S_k. We shall construct inductively a sequence of maps $f_k : M \longrightarrow N$, $k \geq 0$, such that

(1) $f_k(x) = f_r(x)$ for $x \in S_r$, if $r < k$,

(2) f_k is smooth on S_k,

(3) $\rho(f_k(x), f(x)) < \delta(x)$, $x \in M$,

(4) f_k maps C_r into B_r for all k and r.

(Here ρ is a metric on N, and δ is a given positive continuous function on M which we shall adjust for completing the inductive step.)

Define $f_0 = f$, and suppose f_r has been defined for $r \leq k$ satisfying these conditions. Let us write $F = f_k$. Then, since F is smooth on S_k, it is smooth on an open neighbourhood V of S_k. Let $D = C_{k+1} - V \cap C_{k+1}$. Then by (4), $f_k = F$ maps D into B_{k+1}. Choose an open set W in C_{k+1} such that $D \subset W$, and $F(W)$ is contained in B_{k+1}. Since $S_k \subset V$, $D \cap S_k = \emptyset$. Therefore we can find open sets U', V', and U with $\overline{V}'$ compact such that

$$D \subset U',\ \overline{U}' \subset V',\ \overline{V}' \subset U,\ U \subset W,\ U \cap S_k = \emptyset,$$

and U intersects only a finite number of the sets C_r. Since B_{k+1} is a coordinate neighbourhood in N, the map $F|U : U \longrightarrow B_{k+1}$ is given by its components $F^{(i)} : U \longrightarrow \mathbb{R}$, $i = 1, 2, \ldots, \dim N$. Then applying Lemma 3.2 to each component $F^{(i)}$, we get a map $F' : U \longrightarrow B_{k+1}$ such that $|{F'}^{(i)}(x) - F^{(i)}(x)| < \delta(x)$, $x \in U$, for each i, F' is smooth on $V \cup U'$, and $F' = F$ on $U - V'$. Define $f_{k+1} : M \longrightarrow N$ by

$$\begin{aligned} f_{k+1}(x) &= F(x) = f_k(x) \text{ if } x \notin U \\ &= F'(x) \text{ if } x \in U. \end{aligned}$$

Then f_{k+1} satisfies (1), because if $x \in S_r \subset S_k$, $r \leq k$, then $x \notin U$ (as $U \cap S_k = \emptyset$), and so $f_{k+1}(x) = f_k(x) = f_r(x)$. Condition (2) holds, because $f_{k+1} = f_k$ is smooth on S_k, and f' is smooth on $V \cup U'$, which contains $(C_{k+1} \cap V) \cup D = C_{k+1}$. Condition

(3) holds for f_k, and it will hold for f_{k+1} also, because

$$\rho(F'(x), F(x)) = \max\{|F'^{(i)}(x) - F^{(i)}(x)|\} < \delta(x),\ x \in U$$

Condition (4) may be obtained by adjusting the size of δ; note that we need only to impose a finite number of restrictions on δ, since f_{k+1} differs from f_k only on V', and V' intersects only a finite number of the sets C_r.

Having constructed the sequence $\{f_k\}$, define $g : M \longrightarrow N$ by $g(x) = f_k(x)$ for $x \in S_k$. This gives g uniquely by (1), and g is smooth, since f_k is smooth on $\mathrm{Int}S_k$ and these sets cover M. Finally, $g(x) = f_0(x) = f(x)$ for $x \in K \subset S_0$. □

4. Sard's Theorem

Recall that the Lebesgue measure in $\mathbb{R}^n$ is given by a set function

$$\mu : \mathfrak{M} \longrightarrow [0, \infty]$$

satisfying certain axioms, where $\mathfrak{M}$ is a family of certain subsets of $\mathbb{R}^n$ that are called Lebesgue measurable sets. All open, closed, and compact subsets of $\mathbb{R}^n$ are Lebesgue measurable, so are all G_δ and F_σ subsets. We shall use the following properties of the Lebesgue measure: if S and T are Lebesgue measurable sets, then $\mu(S \cup T) \leq \mu(S) + \mu(T)$, and if $S \subset T$, then $\mu(S) \leq \mu(T)$.

An n-dimensional rectangle R in $\mathbb{R}^n$ is the Cartesian product of n intervals $I_1 \times \cdots \times I_n$; it is an n dimensional cube if all the intervals are of equal length. The Lebesgue measure of R is its volume $\mathrm{vol}(R)$ which is the product of the lengths of the n intervals. For an open set U in $\mathbb{R}^n$, $\mathrm{vol}(U) = \inf(\sum_i \mathrm{vol}(Q_i))$, where $\{Q_i\}$ is any sequence of n- dimensional cubes covering U.

A subset K of $\mathbb{R}^n$ has measure zero in $\mathbb{R}^n$ if for any $\epsilon > 0$, K can be covered by a countable collection of n-dimensional cubes such that the sum of their volumes is less than ϵ. This definition may also be given in terms of rectangles, or even n-dimensional balls. A countable union of sets of measure zero has measure zero. For, if $K = K_1 \cup K_2 \cup \cdots$, and $K_i \subset C_i$ where C_i is a countable union of cubes covering K_i and $\mathrm{vol}(C_i) < \epsilon/2^i$, then $K \subset C = \cup C_i$ and $\mathrm{vol}(C) \leq \sum_i \mathrm{vol}(C_i) < \sum \epsilon/2^i = \epsilon$.

The following lemma shows that the condition of being a set of measure zero is invariant under smooth map.

Lemma 4.1. *If a subset A of $\mathbb{R}^n$ has measure zero in $\mathbb{R}^n$, and $f : A \longrightarrow \mathbb{R}^m$ is a smooth map, then $f(A)$ has measure zero in $\mathbb{R}^m$.*

Proof. For each $p \in A$, f has a smooth extension on a neighbourhood of p in $\mathbb{R}^n$, which we still denote by f. By shrinking this neighbourhood, if necessary, we may suppose that f is smooth on a closed n-ball B centered at p. If $u_1, \ldots, u_n$ are the coordinate functions in $\mathbb{R}^n$, then the partial derivatives $\partial f_i/\partial u_j$ are bounded on the compact set B. Then by a theorem of calculus applied to each component f_i of f, we can find a constant c such that

$$\|f(x) - f(y)\| \leq c\|x - y\|$$

for all $x, y \in B$ (see Lemma 5.3 in Part 1). This is called the **Lipschitz estimate** for the smooth map f.

Now given an $\epsilon > 0$, take a countable covering $\{U_j\}$ of $A \cap B$ by open n-balls such that

$$\sum_j \mathrm{vol}(U_j) < \epsilon.$$

Then, by the Lipschitz estimate, $f(B \cap U_j)$ is contained in an n-ball V_j whose radius is not greater than c times the radius of U_j. It follows that $f(B \cap U_j)$ is contained in some of the balls of the collection $\{V_j\}$ whose total volume is not greater than $\sum_j \mathrm{vol}(V_j)$, which is less than $c^n \epsilon$. Since this can be made as small as we like, $f(A \cap B)$ has measure zero. Since $f(A)$ is a union of countably many such sets, it has also measure zero. □

Remark 4.2. The lemma may be false if f is only assumed to be continuous. For example, the subset $A = [0, 1]$ has measure zero in $\mathbb{R}^2$, but there exists a continuous map $f : A \longrightarrow \mathbb{R}^2$ whose image fills up the entire square $[0, 1] \times [0, 1]$, which is not a set of measure zero in $\mathbb{R}^2$.

This is the Hahn-Mazurkiewicz theorem which says that a topological space is a Peano space (i.e. a space which is compact, connected, locally connected,and metric) if and only if it is the image of the unit interval under a continuous map into a Hausdorff space (see Hocking and Young, Topology, p. 129).

Theorem 4.3 (Fubini's Theorem for Measure zero). *If K is a compact set in $\mathbb{R}^n$ such that each subset $K \cap (t \times \mathbb{R}^{n-1})$ has measure zero in the hyperplane $\mathbb{R}^{n-1}$, then K has measure zero in $\mathbb{R}^n$.*

Proof. We may assume that K is contained in the cube I^n, where I is the unit interval $[0, 1]$. Define a function $f : I \longrightarrow \mathbb{R}$ by

$$f(t) = \mu(K \cap ([0, t] \times I^{n-1})), \qquad t \in I,$$

where μ is the Lebesgue measure on $\mathbb{R}^n$. It is required to show that $f(1) = 0$. By hypothesis, given $\epsilon > 0$, there is an open set U in I^{n-1} such that

$$K \cap (t \times I^{n-1}) \subset t \times U \qquad \text{with } \mathrm{vol}(U) < \epsilon.$$

By compactness of K, there is a $h_0 > 0$ such that

$$K \cap ([t - h_0, t + h_0] \times I^{n-1}) \subset [t - h_0, t + h_0] \times U.$$

Then, for any h, $0 \leq h < h_0$,

$$K \cap ([0, t + h] \times I^{n-1}) \subset (K \cap ([0, t] \times I^{n-1})) \cup ([t, t + h] \times U)$$

can be covered by an open set of volume $< f(t) + \epsilon h$. Therefore

$$f(t + h) \leq f(t) + \epsilon h \qquad \text{for } 0 \leq h < h_0.$$

Similarly, we have

$$K \cap ([0, t] \times I^{n-1}) \subset (K \cap ([0, t - h] \times I^{n-1})) \cup ([t - h, t] \times U)$$

so that

$$f(t) \leq f(t - h) + \epsilon h \qquad \text{for } 0 \leq h < h_0.$$

Therefore

$$\left| \frac{f(t + h) - f(t)}{h} \right| \leq \epsilon, \qquad \text{for all } |h| < h_0.$$

Therefore f is differentiable at t and its derivative is zero. Since $f(0) = 0$, we have $f(1) = 0$ also. □

Definition 4.4. A subset K of an n-manifold M is said to have if for each coordinate chart $\phi : U \longrightarrow \mathbb{R}^n$ (or $\mathbb{R}^n_+$) of M, the set $\phi(U \cap K)$ has measure zero in $\mathbb{R}^n$.

It is clear that if $K' \subset K \subset M$ and K has measure zero in M, then K' has also measure zero in M. It is also clear that if $\{K_n\}$ is a countable family of subsets of M such that each K_n has measure zero in M, then $\cup_n K_n$ has measure zero in M.

Let M and N be manifolds of dimension n and m respectively. Then, in view of Definition 4.7 (Part 1), a point $x \in M$ is a critical point of a smooth map $f : M \longrightarrow N$, and $f(x)$ is a critical value of f, provided the Jacobian matrix $Jf(x)$ has rank $< m$. A point $y \in N$ is a regular value of f, if it is not a critical value of f.

By convention, any point of N which is not in $f(M)$ is a regular value of f.

Thus, if $n < m$, then every point of M is a critical point of f, and if $n \geq m$ and $y \in f(M)$ is a regular value of f, then $Jf(x)$ has rank m at every point x of $f^{-1}(y)$.

Theorem 4.5 (Sard). *If $f : M \longrightarrow N$ is a smooth map of manifolds and C is the set of critical points of f in M, then $f(C)$ has measure zero. In other words, almost every point of N is a regular value of f.*

Remark 4.6. A more general version of the theorem, called Morse-Sard theorem, says that if

$$f : M \longrightarrow N$$

is a C^r map, where $\dim M = n$, $\dim N = m$, and $r > \max(0, n-m)$, then the set of critical values of f has measure zero (see Hirsch, Differential Topology, p. 69). The condition on r is necessary, and a counter example (due to Whitney, Duke Math. J. 1 (1935), 514-517) is available, if the inequality be refuted. Whitney constructed a C^1 function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ whose set of critical points C is homeomorphic to the open interval $(0,1) \subset \mathbb{R}$, and $f(C)$ is not a set of measure zero in $\mathbb{R}$. Here $r = \max(0, n-m)$. We will skip the proof of the general version, because Theorem 4.5 is adequate for our purpose.

Lemma 4.7. *If Sard's theorem is true for every smooth map $f : U \longrightarrow \mathbb{R}^m$, where U is an open subset of $\mathbb{R}^n$, then it is also true for every smooth map $g : V \longrightarrow \mathbb{R}^m$, where V is an open subset of $\mathbb{R}^n_+$.*

Proof. Let $g : V \longrightarrow \mathbb{R}^m$, where V is an open subset of $\mathbb{R}^n_+$ be a smooth map, and C be the set of critical points of g. By Lemma 1.14(b), there is an open subset V' of $\mathbb{R}^n$ and a smooth map $g' : V' \longrightarrow \mathbb{R}^m$ such that $V = V' \cap \mathbb{R}^n_+$ and $g'|V = g$. If C' is the set of critical points of g', then $g'(C')$ is a set of measure zero in $\mathbb{R}^m$, by hypothesis. This implies $g(C)$ is a set of measure zero in $\mathbb{R}^m$, because $g(C) = g'(C) \subset g'(C')$. □

Proof of Theorem 4.5. The proof is by induction on n which is the dimension of M. The starting point is $n = 0$ which is trivial. Therefore suppose that the theorem has been proved for all manifolds of dimensions $\leq n-1$. Next note using the Second Axiom of Countability that it suffices to consider only the special case when $f : U \longrightarrow \mathbb{R}^m$, U being an open set of $\mathbb{R}^n_+$, and C is the critical set of f in

U. In view of Lemma 4.7, we may suppose that U is an open set in the interior of $\mathbb{R}^n_+$, or in $\mathbb{R}^n$.

Let D be the set of points in C where the Jacobian matrix $J(f)$ vanishes. We shall show in the next two Lemmas 4.8 and 4.9 that both $f(D)$ and $f(C-D)$ have measure zero in $\mathbb{R}^m$. This will complete the proof of the theorem. □

Lemma 4.8. *The set $f(D)$ has measure zero in $\mathbb{R}^m$.*

Proof. Let $f_1 : U \longrightarrow \mathbb{R}$ be the first component of f. Then, if Jf vanishes at a point x, Jf_1 also vanishes at x, and if K is the set of points where Jf_1 vanishes (K is also the set of critical points of f_1), then $f(D) \subset f_1(K) \times \mathbb{R}^{m-1}$. Therefore if $f_1(K)$ has measure zero in $\mathbb{R}$, then $f_1(K) \times \mathbb{R}^{m-1}$, and hence $f(D)$, has measure zero in $\mathbb{R}^m$, because $\mathbb{R}^{m-1}$ has measure zero in $\mathbb{R}^m$. Hence it is sufficient to prove the lemma for the case $m = 1$.

Let D_i be the set of points of U at which all the partial derivatives of f of order $\leq i$ vanish. We have then a descending sequence of closed subsets of U :

$$D = D_1 \supset D_2 \supset \cdots \supset D_n \supset \cdots .$$

We shall show in the next two sublemmas 1 and 2 that each of the sets $f(D_i - D_{i+1})$, $1 \leq i < n$, and $f(D_n)$ has measure zero. This will complete the proof of the lemma. □

Sublemma 1. *The set $f(D_i - D_{i+1})$, $1 \leq i < n$, has measure zero.*

Proof. It suffices to show that each point p of $D_i - D_{i+1}$, has a neighbourhood V in U such that $f(V \cap (D_i - D_{i+1})$ has measure zero. This will prove that $f(D_i - D_{i+1})$ has measure zero, because $D_i - D_{i+1}$ can be covered by countably many of such neighbourhoods, by the Second Axiom of Countability.

If $p \notin D_{i+1}$, there is an i-th order derivative of f, say g, which vanishes on D_i, but some partial derivative of g say $\partial g/\partial x_1$, is non-zero at p. Define a map $h : U \longrightarrow \mathbb{R}^n$ by $h(x) = (g(x), x_2, \ldots, x_n)$. The Jacobian of h is non-singular at p, and so h maps a neighbourhood V of p diffeomorphically onto an open set W of $\mathbb{R}^n$, by the inverse function theorem. The critical set of $f : V \longrightarrow \mathbb{R}$ is $V \cap (D_i - D_{i+1})$, since $J(f)$ vanishes on D_i. Therefore, since h^{-1} is a diffeomorphism, the critical set of the composition

$$k = f \circ h^{-1} : W \longrightarrow \mathbb{R}$$

is $h(V \cap (D_i - D_{i+1}))$. But $h(V \cap (D_i - D_{i+1})) = (0 \times \mathbb{R}^{n-1})$, and this set is also the critical set of the restriction $k' = k|(0 \times \mathbb{R}^{n-1}) \cap W$. Therefore, by induction (Sard's theorem is true for $n-1$),

$$k'((0 \times \mathbb{R}^{n-1}) \cap W) = f \circ h^{-1}((0 \times \mathbb{R}^{n-1}) \cap W) = f(V \cap (D_i - D_{i+1}))$$

has measure zero. □

Sublemma 2. *The set $f(D_n)$ has measure zero.*

Proof. Again, it will be enough to show that $f(D_n \cap Q)$ has measure for any n-cube Q in U. Let r be the edgelength of Q, and k be a positive integer. Subdivide Q into k^n subcubes of edgelength r/k, and hence of diameter $r\sqrt{n}/k$. Let $p \in D_n \cap Q$, and Q_1 be one of the subcubes containing p. By Taylor's theorem of order n (see the Remark below), if $p+h \in Q_1$, then

$$|f(p+h) - f(p)| \le A \cdot ||h||^{n+1} \le A \cdot (r\sqrt{n}/k)^{n+1},$$

where A is a constant independent of k obtained as a uniform estimate of partial derivatives of f of order $n+1$. Therefore $f(Q_1)$ is contained in an interval of length B/k^{n+1}, where B is a constant independent of k. Hence $f(D_n) \cap Q$ is contained in a union of intervals of total length $\le B \cdot k^n/k^{n+1} = B/k$. Since $\lim_{k\to\infty} B/k = 0$, $f(D_n \cap Q)$ has measure zero. □

Remark. Recall Taylor's theorem of order k in several variables.

Notation. Multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, α_i are integers ≥ 0.
Degree of $\alpha = |\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$.
Monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.
Partial derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

Theorem. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is of class C^{k+1} on an open convex set $U \subset \mathbb{R}^n$. Then if $a, a+h \in U$,

$$f(a+h) = \sum_{|\alpha|\le k} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha + R_{a,k}(h),$$

where the remainder $R_{a,k}(h)$ is given in Lagrange's form as

$$R_{a,k}(h) = \sum_{|\alpha|=k+1} \partial^\alpha f(a+ch) \frac{h^\alpha}{\alpha!},$$

for some c in the unit interval $(0,1)$.

Corollary. If $|\partial^\alpha f(x)| \le M$ for $x \in U$, and $|\alpha| = k+1$, then

$$|R_{a,k}(h)| \le M \sum_{|\alpha|=k+1} \frac{|h^\alpha|}{\alpha!} \le A \cdot \|h\|^{k+1}.$$

Note that $|h^\alpha| = |h_1^{\alpha_1} \cdots h_n^{\alpha_n}| \le \|h\|^{\alpha_1+\cdots+\alpha_n} = \|h\|^{k+1}$, and $\|h\| = \sqrt{h_1^2 + \cdots + h_n^2}$.

Lemma 4.9. *The set $f(C-D)$ has measure zero in $\mathbb{R}^m$.*

Proof. Let $p \notin D$. Then some first order partial derivative of some component of f, say $\partial f_1/\partial x_1$, fails to vanish at p. As in the proof of Sublemma 1, an application of the inverse function theorem asserts that the map $h : U \longrightarrow \mathbb{R}^n$, where $h(x) = (f_1(x), x_2, \ldots, x_n)$, sends a neighbourhood V of p diffeomorphically onto an open set W of $\mathbb{R}^n$. Then the set C_1 of critical points of $g = f \circ h^{-1} : W \longrightarrow \mathbb{R}^m$ is precisely $h(V \cap C)$, and $g(C_1) = f(V \cap C)$.

Now g carries each $(t, x_2, \ldots, x_n) \in W$ into the hyperplane $t \times \mathbb{R}^{m-1} \subset \mathbb{R}^m$. Let $g_t : W \cap (t \times \mathbb{R}^{m-1}) \longrightarrow t \times \mathbb{R}^{m-1}$ be the restriction of g. Since the Jacobian $J(g)$ of g is of the form

$$\begin{pmatrix} 1 & 0 \\ * & J(g_t) \end{pmatrix},$$

a point in $t \times \mathbb{R}^{m-1}$ is a critical point of g_t if and only if it is a critical point of g. By inductive hypothesis, the set of critical values of g_t has measure zero in $t \times \mathbb{R}^{m-1}$. Now Fubini's theorem implies that the set of critical values of g, that is, the set $f(V \cap C)$, is of measure zero. □

Corollary 4.10. *If $\dim M < \dim N$, then a smooth map $f : M \longrightarrow N$ cannot be surjective.*

Proof. The critical set of f is M. Therefore, if f is onto, then $N = f(M)$ will have measure zero, which is not possible. □

Corollary 4.11. *If $f : M \longrightarrow N$ is a smooth map with set of critical points C, then the set $N - f(C)$ is dense in N.*

Proof. A set of measure zero cannot contain a non-empty open set. □

Corollary 4.12. *If $f_i : M \longrightarrow N$ is a countable family of smooth maps, then the set of common regular values of all f_i is dense in N.*

Proof. Any countable union of sets of measure zero has measure zero. □

5. Compact one-manifolds and Brouwer's theorem

We shall show that the only compact connected 1-manifolds are either the closed interval $[0, 1]$ or else the circle S^1, up to diffeomorphism.

Let M be a compact connected 1-manifold. We may suppose that M is a submanifold of $\mathbb{R}^3$. This follows from Whitney Embedding Theorem.

Recall once again that a subset C of M is a **parametrized curve** if it is the image of a diffeomorphism $\phi : I \longrightarrow C$, where I is an interval in $\mathbb{R}$ which may be open, or closed, or half-open (finite or infinite). We shall call ϕ a **parametrizatioin** in M. Its parametric equation is

$$\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t)), t \in I.$$

As t runs over I, $\phi(t)$ traces the curve C. The velocity vector or tangent vector of C at a point $\phi(t)$ is the derivative of $\phi(t)$ at t

$$\phi'(t) = (\phi_1'(t), \phi_2'(t), \phi_3'(t)).$$

As ϕ is a diffeomorphism, $\phi'(t)$ is never zero on I (if $\phi'(t) = 0$, then ϕ cannot be smoothly invertible, by the Inverse Function Theorem).

A **reparametrization** is obtained by a change of parameter $t = t(\theta)$, which is a diffeomorphism $t : J \longrightarrow I$, where J is another interval, as $\psi(\theta) = \phi(t(\theta)), \theta \in J$. Since $dt/d\theta \neq 0$, either $dt/d\theta > 0$ or $dt/d\theta < 0$ on J, that is, t is strictly increasing or strictly decreasing, by Mean Value Theorem. Therefore, if $dt/d\theta > 0$ (resp. < 0) then t increases (resp. decreases) as θ increases, and $\phi(t)$ and $\psi(\theta)$ trace the same curve C in the same (resp. opposite) direction.

The arc length function $s = s(t) = \int_{t_0}^{t} \|\phi'(t)\| dt$ is a change of parameter, since it has continuous non-zero derivative which is the speed function $\|\phi'(t)\|$. The reparametrization $\psi(s) = \phi(t(s))$ is called a parametrization by arc length. Its speed is

$$\|d\psi/ds\| = \|d\phi/dt\| \cdot |dt/ds| = 1.$$

Lemma 5.1. *If $\phi : I \longrightarrow C$ and $\psi : J \longrightarrow D$ are two parametrizations in M such that $C \cap D$ is connected, then $C \cup D$ is a parametrized curve.*

Proof. We may suppose that ϕ and ψ are parametrizations by arc length. Then $\psi^{-1} \circ \phi$ is a diffeomorphism of a relatively open subset I' of I onto a relatively open subset J' of J with constant derivative $+1$ or -1. This means, since $C \cap D$ is connected, that the graph of $\psi^{-1} \circ \phi$ is a straight line segment of slope ± 1 extending from an edge to another edge of the rectangle $I' \times J'$ (note that since the graph is closed and $\psi^{-1} \circ \phi$ is a local diffeomorphism, the graph cannot begin or end in the interior of the rectangle). If $y = \pm x + c$ is the equation of the line segment, then $\psi^{-1} \circ \phi$ may be extended to a diffeomorphism $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ given by $\lambda(s) = \pm s + c$. Now define $\sigma : I \cup \lambda^{-1}(J) \longrightarrow C \cup D$ by $\sigma(s) = \phi(s)$ if $s \in I$, and $\sigma(s) = \psi(\lambda(s))$ if $s \in \lambda^{-1}(J)$. It can be checked easily that σ is well-defined, and it is a diffeomorphism (use chain rule). □

Theorem 5.2. *If M is a compact connected manifold of dimension one, then M is diffeomorphic either to S^1, or to $[0,1]$.*

Proof. Let $\mathcal{S}$ be the family of all pairs (I, ϕ), where $\phi : I \longrightarrow \phi(I) \subseteq M$ is a parametrization. Partially order $\mathcal{S}$ by the binary relation

$$(I, \phi) \leq (J, \psi) \text{ if and only if } I \subseteq J \text{ and } \phi = \psi \mid I.$$

Then any linearly ordered subset $(I_1, \phi_1) \leq (I_2, \phi_2) \leq \cdots$ of $\mathcal{S}$ is bounded by an element $(I, \phi) \in \mathcal{S}$, where I is the interval $\cup_i I_i$. and ϕ is the parametrization given by $\phi \mid I_i = \phi_i$. Therefore by Zorn Lemma, $\mathcal{S}$ contains a maximal parametrized curve C in M and with parametrization $\psi : I \longrightarrow C$. We may suppose by applying a change of variable, if necessary, that I is one of the intervals $(0,1), [0,1), (0,1]$, or $[0,1]$. Then, for any sequence $\{t_n\}$ in I converging to 0 (resp. 1), the sequence $\{\psi(t_n)\}$ in C converges to a unique point x_0 (resp. x_1). It follows that the closure of C is $\overline{C} = C \cup \{x_0, x_1\}$, and ψ extends to a smooth map $\widetilde{\psi} : [0,1] \longrightarrow \overline{C}$ by $\widetilde{\psi}(0) = x_0$ and $\widetilde{\psi}_1(1) = x_1$, of course, $\widetilde{\psi} = \psi$ if x_0 and x_1 are already in C. We consider two cases $x_0 = x_1$, and $x_0 \neq x_1$.

If $x_0 = x_1$, set $\theta = 2\pi s$, where $0 \leq s \leq 1$, define a diffeomorphism $f : S^1 \longrightarrow M$ by

$$f(\cos\theta, \sin\theta) = \widetilde{\psi}(s).$$

Then f is onto, since $f(S^1)$ is compact and open in the connected space M. This proves the first part of the theorem.

If $x_0 \neq x_1$, then the parametrization ψ must map I onto M, that is, C must be equal to M. If this is not true, then any point $x \in M - C$ would admit a parametrized curve neighbourhood D_x not intersecting C, otherwise C would extend to a larger parametrized curve by the above lemma violating the maximality of C. Then C and $\cup_{x \in M - C} D_x$ would give a separation of M, which is not possible since M is connected. □

Corollary 5.3. *The boundary of a compact one- manifold with boundary consists of an even number of points.*

Exercise 5.4. Show that a non-compact connected manifold of dimension one is diffeomorphic to an interval.

Theorem 5.5. *If M is a compact manifold with boundary, then there is no retraction of M onto ∂M.*

Proof. If $f : M \longrightarrow \partial M$ is a retraction, then, by Sard's theorem, there is a point $x \in \partial M$ which is a regular value of f. Then $f^{-1}(x)$ is a submanifold V of M with boundary

$$\partial V = V \cap \partial M.$$

The codimension of V is the codimension of $\{x\}$, which is $n - 1$ if $n = \dim M$. Therefore V is one-dimensional, and it is closed, and so compact. Then ∂V has an even number of points. But, since $f|\partial M = \mathrm{Id}$, ∂V consists of only one point x. This contradiction shows the non-existence of a smooth retraction f. □

Lemma 5.6. *There is no continuous retraction of D^n onto S^{n-1}.*

Proof. Let $f : D^n \longrightarrow S^{n-1}$ be a continuous retraction. Consider the continuous map $g : D^n \longrightarrow D^n$ given by

$$\begin{aligned} g(x) &= 2x, &&\text{if} \quad 0 \le \|x\| \le \tfrac{1}{2} \\ &= x/\|x\|, &&\text{if} \quad \tfrac{1}{2} \le \|x\| \le 1. \end{aligned}$$

Then $h = f \circ g$ is also a retraction, and it is smooth on a closed neighbourhood K of S^{n-1}. The map h can be approximated by a smooth map $k : D^n \longrightarrow S^{n-1}$ which agrees with h on K. This means that k is a smooth retraction, which is in contradiction with the above theorem. □

Theorem 5.7 (Brouwer fixed-point theorem). *Any continuous map f of D^n to itself has a fixed point.*

Proof. Such a continuous map f without fixed point gives rise to a retraction $g : D^n \longrightarrow S^{n-1}$ which sends $x \in D^n$ to a point where the directed line segment from $f(x)$ to x hits the boundary S^{n-1}. We shall show that this map g is continuous. This will be in contradiction with the above lemma, and our proof will be complete.

Since x lies in between $f(x)$ and $g(x)$ on a line segment, we may write

$$g(x) = rx + (1 - r)f(x),$$

where $r \ge 1$. Then g will be continuous, if r is a continuous function of x. Now, since $\|g(x)\| = 1$, the above relation gives a quadratic equation in r

$$r^2\|x - f(x)\|^2 + 2r(x \cdot f(x) - \|f(x)\|^2) + \|f(x)\|^2 - 1 = 0.$$

Solving the quadratic equation, the unique positive root r can be expressed in terms of continuous functions of x. Therefore g is a continuous map, and the proof is complete. □

6. Homotopy of smooth maps

We will now extend the notion of homotopy to the smooth category. Two smooth maps are called smoothly homotopic if one can be deformed to the other through smooth maps. Here is the precise definition.

Definition 6.1. Two smooth maps $f, g : M \longrightarrow N$ are **smoothly homotopic** if there is a smooth map $H : M \times \mathbb{R} \longrightarrow N$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

Thus we have a family of smooth maps $H_t : M \longrightarrow N$ given by $H_t(x) = H(x,t), t \in \mathbb{R}$. The smooth map H is called a **smooth homotopy** between f and g. If H is just a continuous map, then f and g are continuously homotopic, or simply homotopic.

The smooth homotopy is defined for all $t \in \mathbb{R}$, rather than on the interval $I = [0,1]$, because we want to avoid a technical difficulty, namely, $M \times I$ is not a smooth manifold when M has boundary. It may be shown that $M \times I$ can be given a unique smooth structure, using a method called "Smoothing or straightening the corners“. Then we will have no problem in replacing $\mathbb{R}$ by I in the above definition.

The portion of $\mathbb{R}$ outside I does not play any important role. Given H as above, we can always find a smooth map $\overline{H} : M \times \mathbb{R} \longrightarrow N$ such that $\overline{H}(x,t) = f(x)$ if $t \leq 0$ and $\overline{H}(x,t) = g(x)$ if $t \geq 1$. Just define $\overline{H}(x,t) = H(x, \mathcal{B}(t))$, where $\mathcal{B}(t)$ is a bump function (Definition 1.6). The smooth map $\overline{H}$ is called the **normalized homotopy** corresponding to the homotopy H.

Lemma 6.2. *Smooth homotopy is an equivalence relation.*

Proof. That the relation is reflexive and symmetric are obvious. To see that it is transitive, take smooth maps f, g, and h from M to N, and let H and F be normalized smooth homotopies between f and g and between g and h respectively. Define $K : M \times \mathbb{R} \longrightarrow N$ by

$$\begin{aligned} K(x,t) \quad &= H(x,3t) \qquad &\text{if } t \leq 1/2 \\ &= F(x,3t-2) \qquad &\text{if } t \geq 1/2 \end{aligned}$$

This is a smooth map, since H and F are smooth maps and $K(x,t) = g(x)$ for $1/3 \leq t \leq 2/3$ so that two parts of the definition match together smoothly. Clearly K is a normalized homotopy between f and h. □

Lemma 6.3. *If two smooth maps $f, g : M \longrightarrow N$ are continuously homotopic, then they are smoothly homotopic.*

Proof. Let $H : M \times \mathbb{R} \longrightarrow N$ be a normalized continuous homotopy between f and g. Then H is smooth on the closed set $M \times J$, where $J = (-\infty, 0] \cup [1, \infty)$, since $H \mid M \times (-\infty, 0] = f$ and $H \mid M \times [1, \infty) = g$. By Theorem 2.3, there is a positive continuoud function δ on M such that H can be δ-approximated by a smooth map $F : M \times \mathbb{R} \longrightarrow N$ which agrees with H on $M \times J$. □

Exercise 6.4. Show that if m is sufficiently large, then any smooth map

$$f : M \longrightarrow \mathbb{R}^m$$

is δ-approximable by an embedding $g : M \longrightarrow \mathbb{R}^m$ which is homotopic to f by a smooth homotopy $H_t : M \longrightarrow \mathbb{R}^m$ so that each H_t is a δ-approximation to f.

Hint. $H_t(x) = (1-t)f(x) + tg(x)$.

Definition 6.5. Two embeddings $f, g : M \longrightarrow N$ are **isotopic** if there exists a smooth homotopy $H : M \times \mathbb{R} \longrightarrow N$ such that for each $t \in \mathbb{R}$, the map

$$H_t : M \longrightarrow N$$

is an embedding.

Remark 6.6. If $H_t : M \longrightarrow N$ is an isotopy, and

$$\alpha : M_1 \longrightarrow M, \quad \beta : N \longrightarrow N_1$$

are embeddings, then $\beta \circ H_t \circ \alpha : M_1 \longrightarrow N_1$ is an embedding.

Proposition 6.7. *Any two embeddings $f, g : M \longrightarrow \mathbb{R}^m$ are isotopic, provided m is sufficiently large (in fact, $m \geq 2n + 2$, where $n = \dim M$).*

Proof. Since $\mathbb{R}^m$ is contractible to a point, the embeddings f and g are continuously homotopic, and hence homotopic by a smooth homotopy $H : M \times \mathbb{R} \longrightarrow \mathbb{R}^m$. If m is sufficiently large, H may be deformed to an embedding $F : M \times \mathbb{R} \longrightarrow \mathbb{R}^m$ which agrees with H on $(-\infty, 0] \cup [1, \infty)$. This F serves as the required isotopy between f and g. □

7. Stability of smooth maps

We now consider a different set of problems. Suppose, for example, an embedding f is deformed slightly to a map g; then we would like to pose the question whether g also an embedding.

Definition 7.1. Let $\mathcal{C}$ be a class of smooth maps from M to N defined by a property. Then $\mathcal{C}$ is called a **stable class** with respect to the property if for any $f \in \mathcal{C}$ and any smooth homotopy $f_t : M \longrightarrow N$ of f, there is an $\epsilon > 0$ such that $f_t \in \mathcal{C}$ for all $t < \epsilon$.

Theorem 7.2. *Each of the following classes of smooth maps from M to N, where M is compact and $\partial M = \partial N = \emptyset$, is a stable class:*

(1) *local diffeomorphisms,*
(2) *immersions,*
(3) *submersions,*
(4) *embeddings,*
(5) *diffeomorphisms.*

To this list of classes of maps, we may add one more class, namely, the class of maps transvesal to a given submanifold A of N. We will read about this class of maps in Part 3, and show that locally the transversality condition is the same as the submersion condition (3). Therefore this class will be stable.

Proof. We shall prove only (2) and (4). Because, (1) is a special case of (2) when $\dim M = \dim N$, and the proof of (3) is essentially identical with the proof of (2). The proof of (5) will follow from (4) and the fact that a local diffeomorphism maps open sets into open sets.

Proof of (2). Let f_t be a smooth homotopy of an immersion f_0. Then the problem is to find an $\epsilon > 0$ so that $d(f_t)_x$ is injective for all points

$$(x, t) \in M \times [0, \epsilon) \subset M \times I.$$

Since M is compact, any open neighbourhood of $M \times \{0\}$ in $M \times I$ contains $M \times [0, \epsilon)$ if ϵ is small enough. Therefore, it is sufficient only to show that each point $(x_0, 0) \in M \times \{0\}$ has an open neighbourhood U in $M \times I$ such that $d(f_t)_x$

is injective for $(x,t) \in U$. Since this assertion is local, it is enough to consider only the case when M is an open subset of $\mathbb{R}^n$, and N is an open subset of $\mathbb{R}^m$.

Since $d(f_0)_{x_0}$ is injective, the Jacobian matrix $Jf_0(x_0)$ of f_0 at x_0 has a minor $R(x_0,0)$ of order n whose determinant is non-zero. The function $M \times I \longrightarrow \mathbb{R}$, which sends (x,t) to the determinant of the minor $R(x,t)$ of the Jacobian matrix $Jf_t(x)$ (formed by the same rows and columns as $R(x_0,0)$) is continuous, since each entry of $R(x,t)$ is continuous on $M \times I$, and the determinant function is continuous. Therefore there is an open neighbourhood U of $(x_0,0)$ in $M \times I$ such that $R(x,t)$ is non-singular for all $(x,t) \in U$. This completes the proof of (2).

Proof of (4). As shown in the above proof, f_0 is an immersion implies that f_t is an immersion for small values of t. We shall show that if f_0 is injective, then so is f_t for sufficiently small t. This will complete the proof of (4), because any injective immersion on a compact manifold is an embedding.

Suppose our assertion is false. Take a sequence of real numbers $\{t_k\}$ which converges to zero. For each k, we can find a pair of distinct points (x_k, y_k) of M such that $f_{t_k}(x_k) = f_{t_k}(y_k)$. Since M is compact, each of the sequences $\{x_k\}$ and $\{y_k\}$ has convergent subsequences. Denoting them by the same notations, let $\lim x_k = x_0$ and $\lim y_k = y_0$. Then

$$f_0(x_0) = \lim f_{t_k}(x_k) = \lim f_{t_k}(y_k) = f_0(y_0).$$

This implies that $x_0 = y_0$, since f_0 is injective.

Define a smooth map $G : M \times I \longrightarrow N \times I$ by $G(x,t) = (f_t(x), t)$ A simple computation shows that the Jacobian matrix $JG(x_0,0)$ is

$$\begin{pmatrix} \boxed{Jf_0(x_0)} & * \\ 0, \cdots, 0 & 1 \end{pmatrix},$$

which is non-singular, since $Jf_0(x_0)$ is so. Then, by the inverse function theorem, G is injective in a neighbourhood of $(x_0,0)$. But for large k, both (x_k,t_k) and (y_k,t_k) belong to this neighbourhood, and so $x_k = y_k$, which is a contradiction. Therefore we may conclude that f_t is injective when t is sufficiently small. □

Exercise 7.3. Show that the theorem is false if M is not compact, by constructing counterexamples to all the classes in the following way:

Define $f_t : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_t(s) = s\lambda(ts)$, where $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function with $\lambda(s) = 1$ if $|s| < 1$, and $\lambda(s) = 0$ if $|s| > 2$.