AFS-II Almora, May 1 - 28, 2014

Lecture on Differential Topology Part I

Speakers: Samik Basu (RKMVU, Calcutta), Goutam Mukherjee (ISI, Calcutta), Amiya Mukherjee (ISI, Calcutta)

1. Smooth manifolds

Intuitively a differentiable manifold is a topological space which is obtained by gluing together open subsets of some Euclidean space in a nice way; think, for example, of the surface of a ball or a torus covered with small paper disks pasted together on overlaps without making any crease or fold. Mathematical definition is based on the standard differentiable structure on a Euclidean space $\mathbb{R}^n$. Let $u_1, \dots, u_n$ denote the coordinate functions, where $u_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the function mapping a point $p = (p_1, \dots, p_n)$ onto its i-th coordinate p_i. A function f from an open subset U of $\mathbb{R}^n$ into $\mathbb{R}$ is **differentiable of class** C^r, or simply a C^r **function**, if it has continuous partial derivatives of all orders $\leq r$ with respect to $u_1, \dots, u_n$. A C^0 **function** is just a continuous function. A C^∞ **function** is C^r for every $r \geq 0$.

A map $\phi : U \longrightarrow \mathbb{R}^m$, U open in $\mathbb{R}^n$, can be written as $\phi = (\phi_1, \dots, \phi_m)$, where $\phi_i = u_i \circ \phi : U \longrightarrow \mathbb{R}$ are the components of ϕ. The map ϕ is C^r if each ϕ_i is C^r. A map ϕ between two open subsets of $\mathbb{R}^n$ is called a C^r **diffeomorphism** if it is a homeomorphism and both ϕ and ϕ^{-1} are C^r maps. We shall call a C^∞ diffeomorphism simply a diffeomorphism. For example, any linear isomorphism $R^n \longrightarrow \mathbb{R}^n$ is a diffeomorphism.

We shall use the words “smooth”, “differentiable”, and the symbol “C^∞” interchangeably. Our standard practice in this lecture will be to work with smooth maps.

Definition 1.1. A **smooth manifold** M of dimension n is a second countable Hausdorff space together with a smooth structure on it. A **smooth structure** consists of a family $\mathcal{D}^\infty$ of pairs (U_i, ϕ_i), i is in some index set Λ, where U_i is an open set of M and ϕ_i is a homeomorphism of U_i onto an open set of $\mathbb{R}^n$ such that

(1) the open sets U_i, $i \in \Lambda$, cover M,
(2) for every pair of indices $i, j \in \Lambda$ with $U_i \cap U_j \neq \emptyset$ the homeomorphisms
$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j),$$
$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$
are smooth maps between open subsets of $\mathbb{R}^n$,
(3) the family $\mathcal{D}^\infty$ is maximal in the sense that it contains all possible pairs (U_i, ϕ_i) satisfying the property (2).

The restriction $U_i \cap U_j \neq \emptyset$ in the condition (2) may be omitted provided we agree to assume that the empty map on the empty set is smooth.

A pair $(U, \phi) \in \mathcal{D}^\infty$ with $p \in U$ is called a **coordinate chart** at p, U is called a **coordinate neighbourhood** of p, and $\phi = (x_1, \dots, x_n)$, where $x_i = u_i \circ \phi : U \longrightarrow \mathbb{R}$ is the i-th component of ϕ, is called a (local)**coordinate system** at p.

Two charts (U_i, ϕ_i) and (U_j, ϕ_j) satisfying the conditions in (2) are said to be C^∞ **related** or **compatible**, and each of $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ is called a **transition map** or a **change of coordinates**. A family of coordinate charts on M satisfying (1) and (2) is called a **smooth atlas** [1]. A smooth structure is a smooth atlas satisfying (3).

To understand the maximality condition (3) more clearly, consider the family of all smooth atlases on M. Say that two atlases $\mathcal{A}$ and $\mathcal{B}$ are compatible if each chart in $\mathcal{A}$ is compatible with each chart in $\mathcal{B}$, or equivalently, if $\mathcal{A} \cup \mathcal{B}$ is an atlas on M. It is easy to check that this is an equivalence relation. Then the union of all atlases in an equivalence class is a maximal atlas or a smooth structure on M. Thus any atlas can be enlarged to a unique smooth structure by adjoining all smoothly related charts to it.

The maximality condition allows us to restrict coordinate charts. If (U, ϕ) is a chart, U' is an open set in U, and $\phi' = \phi|U'$, then the charts (U, ϕ) and (U', ϕ') are compatible by the transition map $\phi' \circ \phi^{-1} = \mathrm{id}$, where id denotes the identity map.

Next observe that the charts (U, ϕ) and $(U, \alpha \circ \phi)$, where $\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a diffeomorphism, are always compatible. In particular, taking α to be the translation which sends $\phi(p)$ to 0, we can always suppose that every point $p \in M$ admits a coordinate chart (U, ϕ) such that $\phi(p) = 0$. We may also suppose that $\phi(U)$ is a convex set, or the whole of $\mathbb{R}^n$.

Examples 1.2. (1) **Euclidean space** $\mathbb{R}^n$. A smooth structure is given by an atlas consisting of only one chart $(\mathbb{R}^n, id)$. The maximal atlas generated by this atlas consists of all charts (U, ϕ), where U is an open set of $\mathbb{R}^n$ and ϕ is Id on it. This smooth structure on $\mathbb{R}^n$ is called the **standard smooth structure**.

A similar consideration shows that the complex n-space $\mathbb{C}^n$ is a smooth complex manifold of complex dimension n.

(2) **Vector space**. Any real vector space V of dimension n has a canonical smooth structure generated by the atlas consisting of all linear isomorphisms of V onto $\mathbb{R}^n$. Note that in this atlas any change of coordinates is a linear map and so indefinitely differentiable.

(3) **Open subset of a smooth manifold**. An open set V of a smooth manifold M is itself a smooth manifold. The smooth structure is obtained by restrictions of coordinate charts. If $\mathcal{A}$ is a smooth atlas for M, then $\mathcal{A}_V = \{(U \cap V, \phi|U \cap V) : (U, \phi) \in \mathcal{A}\}$ is a smooth atlas for V.

(4) **Manifold of matrices**. Let $\mathbb{K}$ denote the field $\mathbb{R}$ or $\mathbb{C}$, and $M(m, n, \mathbb{K})$ be the space of all $m \times n$ matrices with entries in $\mathbb{K}$. Taking the entries of matrices in lexicographic (or dictionary) order we may identify $M(m, n, \mathbb{K})$ with $\mathbb{K}^{mn}$ in the following way:

$$(a_{ij}) \leftrightarrow (a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, a_{22}, \ldots, a_{2n}, \ldots, a_{m1}, a_{m2}, \ldots, a_{mn}).$$

Thus $M(m, n, \mathbb{R})$ is a smooth manifold of dimension mn, and, similarly $M(m, n, \mathbb{C})$ is a smooth complex manifold of real dimension $2mn$.

---

[1] The terminology is probably due to Carl Friedrich Gauss (1777-1855) who formulated in mathematical terms the method of drawing maps of earth's surface.

(5) **General linear group** $GL(n, \mathbb{K})$. If $n = m$, let us write the manifold of matrices $M(n, n, \mathbb{K})$ as $M(n, \mathbb{K})$. Then, the set $GL(n, \mathbb{K})$ of all non-singular matrices of order n forms an open subset of $M(n, \mathbb{K})$, since the determinant function $\det : M(n, \mathbb{K}) \longrightarrow \mathbb{K}$ is continuous, being a polynomial map. Therefore $GL(n, \mathbb{K})$ is a smooth manifold.

(6) **Sphere** S^n. This is the set of all unit vectors in $\mathbb{R}^{n+1}$

$$S^n = \{(x_1, \dots, x_{n+1}) \in R^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}.$$

A smooth atlas is provided by two open sets U_+ and U_- obtained by deleting from S^n the north pole $P = (0, \dots, 0, 1)$ and the south pole $Q = (0, \dots, 0, -1)$ respectively, and the stereographic projections

$$\phi_+ : U_+ \longrightarrow \mathbb{R}^n, \text{ and } \phi_- : U_- \longrightarrow \mathbb{R}^n$$

from P and Q onto the equatorial plane $x_{n+1} = 0$. These are homeomorphisms given by

$$\phi_\pm(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1 \mp x_{n+1}}, \dots, \frac{x_n}{1 \mp x_{n+1}} \right),$$

and their inverses are

$$(\phi_\pm)^{-1}(x_1, \dots, x_n) = \left(\frac{2x_1}{1 + \|x\|^2}, \dots, \frac{2x_n}{1 + \|x\|^2}, \mp \frac{1 - \|x\|^2}{1 + \|x\|^2} \right).$$

Therefore the change of coordinates $\phi_- \circ \phi_+^{-1} = \phi_+ \circ \phi_-^{-1} : \mathbb{R}^n - \{0\} \longrightarrow \mathbb{R}^n - \{0\}$ is given by the smooth map $x \mapsto x / \parallel x \parallel^2$.

Exercise 1.3. Show that another smooth atlas of S^n is given by the $2n + 2$ coordinate charts (V_i^+, ψ_i^+), (V_i^-, ψ_i^-), $i = 1, \dots, n + 1$, where V_i^+ and V_i^- are the hemispheres

$$V_i^+ = \{x \in S^n : x_i > 0\}, \qquad V_i^- = \{x \in S^n : x_i < 0\}$$

and $\psi_i^+ : V_i^+ \longrightarrow \mathbb{R}^n$ and $\psi_i^- : V_i^- \longrightarrow \mathbb{R}^n$ are the projections onto the hyperplane $x_i = 0$

$$(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}).$$

Show that these charts are C^∞ related to the charts (U_+, ϕ_+) and (U_-, ϕ_-) of Example 1.2 (6).

Example 1.4. The real projective space $\mathbb{R}P^n$ This space is the quotient space of $\mathbb{R}^{n+1} - \{0\}$ modulo the equivalence relation:

$$(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n), \quad \lambda \in \mathbb{R} - \{0\}.$$

The equivalence classes are 1-dimensional subspaces or lines through the origin in $\mathbb{R}^{n+1}$. Let $\pi : \mathbb{R}^{n+1} - \{0\} \longrightarrow \mathbb{R}P^n$ be the canonical projection, which maps a point x to the line containing x, and let $\mathbb{R}P^n$ be given the quotient topology so that π becomes a continuous open map.

For each i, $0 \le i \le n$, consider open subset U_i of $\mathbb{R}P^n$ given by

$$U_i = \{[x_0, \dots, x_n] \mid x_i \neq 0\},$$

where $[x_0, \dots, x_n] = \pi((x_0, \dots, x_n))$. This is the set of all lines through the origin which intersect the hyperplane $x_i = 1$, and this is open in $\mathbb{R}P^n$ because

$$\pi^{-1}(U_i) = \mathbb{R}^{n+1} - \{\text{hyperplane } x_i = 0\}$$

is open in $\mathbb{R}^{n+1} - \{0\}$. Define $\phi_i : U_i \longrightarrow \mathbb{R}^n$ by

$$\phi_i([x_0, \dots, x_n]) = \frac{1}{x_i}(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

Then ϕ_i is a homeomorphism with inverse given by

$$\phi_i^{-1}(x_1, \dots, x_n) = [x_1, \dots, x_i, 1, x_{i+1}, \dots, x_n].$$

So the change of coordinates between charts (U_i, ϕ_i) and (U_j, ϕ_j) is

$$\phi_j \circ \phi_i^{-1}(x_1, \dots, x_n) = \frac{1}{x_{j+1}}(x_1, \dots, x_j, x_{j+2}, \dots, x_i, 1, x_{i+1}, \dots, x_n),$$

assuming for convenience $j < i$. This the family $\{(U_i, \phi_i)\}$ is a smooth atlas for $\mathbb{R}P^n$.

Exercise 1.5. Complex projective space $\mathbb{C}P^n$. This is the set of all 1-dimensional complex linear subspaces of $\mathbb{C}^{n+1}$ with the quotient topology obtained from the natural projection $\pi : \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{C}P^n$. Show that this can be given a smooth structure analogous to above construction for $\mathbb{R}P^n$

Example 1.6. Product of manifolds. If M and N are smooth manifolds with smooth structures $\{(U_i, \phi_i)\}$ and $\{(V_r, \psi_r)\}$ respectively, then the Cartesian product $M \times N$ is a smooth manifold with atlas $\{(U_i \times V_r, \phi_i \times \psi_r)\}$. Any two such charts are smoothly compatible, because

$$(\phi_j \times \psi_s) \circ (\phi_i \times \psi_r)^{-1} = (\phi_j \times \psi_s) \circ (\phi_i^{-1} \times \psi_r^{-1}) = (\phi_j \circ \phi_i^{-1}) \times (\psi_s \circ \psi_r^{-1}),$$

which is a smooth map.

In particular, the n-torus $T^n = S^1 \times \cdots \times S^1$ (S^1 appearing n times) is a smooth manifold.

2. Smooth map between manifolds

Let M and N be smooth manifolds, and $f : M \to N$ a map. Let $p \in M$, and (U, ϕ) and (V, ψ) be coordinate charts at p and $f(p)$ respectively Then the map $\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \longrightarrow \psi(V)$ is called a **local representation** of f at p for the pair of coordinate systems (ϕ, ψ).

Definition 2.1. A map $f : M \longrightarrow N$ is **smooth**, if its local representation at every point $p \in M$ is a smooth map for some, and hence for all pairs of coordinate systems ϕ and ψ at p and at $f(p)$.

Observe that this definition is independent of the choice of coordinate systems. If f is smooth at p for a pair (ϕ, ψ), then it is smooth at p for every other pair (ϕ_1, ψ_1). Because, the transition maps $\phi \circ \phi_1^{-1}$ and $\psi \circ \psi_1^{-1}$ are smooth, and so the composition

$$\psi_1 \circ f \circ \phi_1^{-1} = (\psi_1 \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \phi_1^{-1})$$

is smooth.

Lemma 2.2. *The composition of smooth maps between manifolds is smooth.*

Proof. For suitable coordinate charts (U,ϕ), (V,ψ), and (W,θ) in M, N, and R respectively, the map

$$\theta \circ (g \circ f) \circ \phi^{-1} = (\theta \circ g \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1})$$

is smooth, being the composition of smooth maps between open subsets of Euclidean spaces. □

Definition 2.3. A map $f : M \longrightarrow N$ is called a **diffeomorphism** if f is a bijection and both f and f^{-1} are smooth maps.

For example, if (U,ϕ) is a coordinate chart on M, then $\phi : U \longrightarrow \mathbb{R}^n$ is a diffeomorphism onto its image, since its local representation for the pair of charts (U,ϕ) and $(\phi(U),\ \mathrm{id})$ is the identity map.

Smooth maps are defined on open subsets of a manifold. The definition can be extended over arbitrary subsets of a manifold in the following way.

Definition 2.4. A map f from a subset S of a manifold M to a manifold N is **smooth** if it can be locally extended to a smooth map. Explicitly, f is smooth, if each point $p \in S$ admits an open neighbourhood U in M and a smooth map $F : U \longrightarrow N$ such that $F|S \cap U = f$.

The local extendability condition of f is equivalent to saying that all the partial derivatives of f exist and are continuous, by Whitney's extension theorem (Whitney, Trans. Amer. Math. Soc. 36 (1936), 63-89).

Exercise 2.5. Show that if $n < m$, and $\mathbb{R}^n$ is considered as the subset

$$\{(x_1, \ldots, x_m) \mid x_{n+1} = \cdots = x_m = 0\}$$

of the first n coordinates of $\mathbb{R}^m$, then the usual smooth maps on $\mathbb{R}^n$ and those obtained by using the above definition are the same.

A map f from a subset S of a manifold M to a subset K of a manifold N is a **diffeomorphism** if it is a bijection and both f and f^{-1} are smooth maps.

It follows that a subset S in an Euclidean space $\mathbb{R}^m$ is a smooth manifold of dimension n if it is locally diffeomorphic to $\mathbb{R}^n$, that is, if each point of S has an open neighbourhood in S (in the relative topology) which is diffeomorphic to an open subset of $\mathbb{R}^n$. Here is an example.

Example 2.6. Space of matrices of rank k. Let $M_k(m,n,\mathbb{R})$ be the space of all real $m \times n$ matrices of rank k, where $0 < k \leq \min(m,n)$, with the induced topology of $M(m,n,\mathbb{R})$. Then $M_k(m,n,\mathbb{R})$ is a smooth manifold of dimension $k(m+n-k)$. To see this, take an element $E_0 \in M_k(m,n,\mathbb{R})$. We may assume by permuting the rows and columns, if necessary, that E_0 is of the form

$$E_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix},$$

where A_0 is a non-singular $k \times k$ matrix. Then, we can find an $\epsilon > 0$ such that if A is a $k \times k$ matrix and if each entry of $A - A_0$ has absolute value less than ϵ, then A is non-singular. Let

$$U = \left\{ E = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid \text{absolute values of all entries of } A - A_0 < \epsilon \right\}.$$

A matrix E as above has the same rank as the matrix

$$\begin{pmatrix} I_k & 0 \\ X & I_{m-k} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ XA + C & XB + D \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix and X is any $(m-k) \times k$ matrix. Taking $X = -CA^{-1}$, we find that the rank of E is exactly k if and only if $D = CA^{-1}B$. Let V be the open set in the Euclidean space of dimension $mn - (m-k)(n-k) = k(m+n-k)$ consisting of matrices of the form

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix},$$

where each entry of $A - A_0$ has absolute value less than ϵ. Then the map

$$\begin{pmatrix} A & B \\ C & CA^{-1}B \end{pmatrix} \longrightarrow \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

is a diffeomorphism of the neighbourhood $U \cap M_k(m, n, \mathbb{R})$ of E_0 onto V. Since E_0 is an arbitrary element of $M_k(m, n, \mathbb{R})$, $M_k(m, n, \mathbb{R})$ is a smooth manifold of dimension $k(m+n-k)$.

Exercise 2.7. Show that if M and N are smooth manifolds, and there is a diffeomorphism of M onto a subset S of N, then S is a smooth manifold.

Exercise 2.8. The graph of a map $f : M \longrightarrow N$ is the set

$$\Gamma(f) = \{(x, f(x)) \in M \times N \mid x \in M\}.$$

Show that if f is smooth, then the map $F : M \longrightarrow \Gamma(f)$ defined by $F(x) = (x, f(x))$ is a diffeomorphism. Conclude that $\Gamma(f)$ is a smooth manifold. In particular, the diagonal set $\triangle$ in $M \times M$, which is $\Gamma(\mathrm{Id}_M)$, is a smooth manifold.

3. Immersions and Submersions

Convention. From now on, by a manifold we shall always mean a smooth manifold, unless it is stated explicitly otherwise. Sometimes we call a manifold M of dimension n an n-manifold, if it be necessary to specify its dimension.

We recall from calculus the process of derivation which assigns to each differentiable map and each point of its domain a linear map.

Definition 3.1. Let $U \subset \mathbb{R}^n$ be an open set, and $a \in U$. Then a map $f : U \longrightarrow \mathbb{R}^m$ is **differentiable** at a if there is a linear map $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$\lim_{u \to a} \frac{\|f(u) - f(a) - L(u-a)\|}{\|u - a\|} = 0.$$

The linear map L is unique. For, if $L' : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is another such linear map, then we have for $v \neq 0$

$$\frac{\|L(v) - L'(v)\|}{\|v\|} = \lim_{t \to 0} \frac{\|L(tv) - L'(tv)\|}{\|tv\|}$$

$$\leq \lim_{t \to 0} \frac{\|f(a+tv) - f(a) - L(tv)\|}{\|tv\|} + \lim_{t \to 0} \frac{\|f(a+tv) - f(a) - L'(tv)\|}{\|tv\|} = 0,$$

and so $L(v) = L'(v)$ for all $v \in \mathbb{R}^n$.

The linear map L is called the **derivative map** (or **total derivative**) of f at a, and is denoted by $df_a : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Its value at $v \in \mathbb{R}^n$ is given by

$$df_a(v) = \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t}.$$

For future reference, we list some well-known results.

Proposition 3.2. *The derivative map enjoys the following properties.*

(1) *If df_a exists, then f is continuous at a.*
(2) *If f is a constant map, then $df_a = 0$.*
(3) *If f is a linear map, then $df_a = f$.*
(4) *If $f, g : U \longrightarrow \mathbb{R}^m$ are differentiable at a, then $f + g$ is differentiable at a, and $d(f + g)_a = df_a + dg_a$.*
(5) *If $\lambda : U \longrightarrow \mathbb{R}$ and $f : U \longrightarrow \mathbb{R}^m$ are differentiable at a, then λf is differentiable at a, and $d(\lambda f)_a = \lambda(a)df_a + f(a)d\lambda_a$.*
(6) (Chain Rule). If $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ are open sets, and $f : U \longrightarrow V$, $g : V \longrightarrow \mathbb{R}^p$ are differentiable maps, then their composition $g \circ f$ is differentiable, and, for each $a \in U$,

$$d(g \circ f)_a = dg_{f(a)} \circ df_a.$$

If $m = 1$, and $(\alpha_1, \ldots, \alpha_n)$ is an orthonormal basis of $\mathbb{R}^n$ with coordinate functions $u_1, \ldots, u_n$ so that, for $p \in \mathbb{R}^n$, $u_i(p) = \langle p, \alpha_i \rangle$ is the i-th coordinate of p, then $df_a(\alpha_i)$ is the i-th partial derivative $\partial f / \partial u_i(a)$ of f at a. Setting $v = v_1\alpha_1 + \cdots + v_n\alpha_n$, we have

$$df_a(v) = v_1 \frac{\partial f}{\partial u_1}(a) + \cdots + v_n \frac{\partial f}{\partial u_n}(a),$$

by the properties (2), (4), and (5).

In general, if $(\beta_1, \ldots, \beta_m)$ is an orthonormal basis of $\mathbb{R}^m$ so that

$$f(u) = \sum_{i=1}^{m} f_i(u)\beta_i,$$

where the components $f_i : U \longrightarrow \mathbb{R}$ are continuous and satisfy $f_i(u) = \langle f(u), \beta_i \rangle$, then df_a exists if and only if df_{ia} exists, and in that case

$$df_a(v) = \sum_{i=1}^{m} df_{ia}(v)\beta_i = \sum_{i=1}^{m} \left(v_1 \frac{\partial f_i}{\partial u_1}(a) + \cdots + v_n \frac{\partial f_i}{\partial u_n}(a) \right) \beta_i.$$

It follows that the matrix of the linear map df_a with respect to the bases α_i and β_j is the Jacobian matrix

$$Jf(a) = \left(\frac{\partial f_i}{\partial u_j}(a) \right).$$

Note that $f : U \longrightarrow \mathbb{R}^m$ is a C^1-map if and only if the map $df : U \longrightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ sending a to df_a, where $L(\mathbb{R}^n, \mathbb{R}^m)$ denotes the vector space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, is continuous.

Let f be a smooth function from an open set V of an n-manifold M into $\mathbb{R}$. Then, for every chart (U, ϕ) on M with $U \cap V \neq \emptyset$, the function $f \circ \phi^{-1} : \phi(U \cap V) \longrightarrow \mathbb{R}$

is smooth. If $\phi = (x_1, \dots, x_n)$, $x_i = u_i \circ \phi$, then the **partial derivative** of f with respect to x_i at $p \in U \cap V$, is defined by

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial (f \circ \phi^{-1})}{\partial u_i}(\phi(p)).$$

Let M and N be manifolds of dimension n and m respectively. If $f : M \longrightarrow N$ is a smooth map, and $\phi = (x_1, \dots, x_n)$ and $\psi = (y_1, \dots, y_m)$ are coordinate systems in M and N respectively, then the functions $f_i = y_i \circ f$ of $x_1, \dots, x_n$ are called the **components** of f. The **Jacobian matrix** of f relative to the pair of coordinate systems (ϕ, ψ) is defined to be the $m \times n$ matrix

$$Jf = \left(\frac{\partial f_i}{\partial x_j} \right).$$

Note that this is nothing but the Jacobian matrix Jg of the local representation $g = \psi \circ f \circ \phi^{-1}$. The **rank** of f at p is defined to be the rank of $Jf(p)$. The definition is independent of the local representation of f. This may be seen easily. Suppose that $g = \psi \circ f \circ \phi^{-1}$ and $g' = \psi' \circ f \circ \phi'^{-1}$ are two local representations of f at p for the pairs of coordinate charts (U, ϕ), (V, ψ) and (U', ϕ'), (V', ψ') respectively. We may suppose that $U = U'$ and $V = V'$, by replacing U, U' by $U \cap U'$ and V, V' by $V \cap V'$. Then $g' = (\psi' \circ \psi^{-1}) \circ g \circ (\phi \circ \phi'^{-1})$. This proves the assertion, since $\phi \circ \phi'^{-1}$ and $\psi' \circ \psi^{-1}$ are diffeomorphisms.

We will now prove some theorems which will provide the keys to understanding the local behaviour of a smooth map of maximum rank.

Theorem 3.3 (**Inverse Function Theorem**). *Let M and N be manifolds of the same dimension n, and $f : U \longrightarrow V$ be a smooth map, where U and V are open sets of M and N respectively. Then, if rank $f = n$ at a point $p \in U$, there exists an open neighbourhood W of p in U such that $f|W$ is a diffeomorphism onto an open neighbourhood of $f(p)$ in V.*

Proof. The theorem is just the Inverse Function Theorem of Calculus when $M = N = \mathbb{R}^n$, and its proof follows trivially from this special case. By hypothesis, any local representation $g = \psi \circ f \circ \phi^{-1}$ of f has rank n at the point $\phi(p)$, and therefore there is an open neighbourhood W' of $\phi(p)$ on which g is a diffeomorphism. Then the restriction of f to $W = \phi^{-1}(W')$ is also a diffeomorphism. □

The next theorem generalizes this result, when $\dim M \leq \dim N$.

Definition 3.4. Let M and N be manifolds of dimension n and m respectively. A smooth map $f : M \longrightarrow N$ is called an *immersion* at $p \in M$ if $n \leq m$ and rank $f = n$ at p. It is called a *submersion* at p if $n \geq m$ and rank $f = m$ at p. The map f is called an **immersion**, or a **submersion**, if it is so at each point of M.

Also, f is called an **embedding** if it an immersion, and a homeomorphism onto its image $f(M)$. If $n = m$, then a surjective embedding is a diffeomorphism.

.

Examples 3.5. (1) If $n \leq m$, the standard inclusion map $i : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0)$ is an embedding. It is called the **canonical embedding**.

(2) If $n \geq m$, the projection map $s : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ onto the first m coordinates given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m)$ is a submersion. It is called the **canonical submersion**.

The following examples show that an injective immersion may not be an embedding.

Example 3.6. The map $f : [0, 2\pi] \longrightarrow \mathbb{R}^2$ given by $f(t) = (\sin 2t, -\sin t)$ is an immersion. As t varies from 0 to 2π, the image point traces the lower half of the figure "8″ in the clockwise direction, and then traces the upper half in the anti-clockwise direction. (The Cartesian equation of the curve is $x^2 = 4y^2(1-y^2)$.) It is not an embedding, because there is a crossing at the origin. The restriction $f|(0, 2\pi)$ is an injective immersion, but not an embedding, as it is not a homeomorphism onto its image (the ends are not joined). However, the restriction $f|(0, \pi)$ is a embedding, as the image is the lower half of the figure '8' without the origin.

Example 3.7. Consider the map $f : \mathbb{R} \longrightarrow S^1 \times S^1$ given by

$$f(t) = (e^{2\pi i\alpha t}, e^{2\pi i\beta t}),$$

where α/β is irrational, The map is an immersion, since df/dt is never zero. It is injective, since $f(t_1) = f(t_2)$ implies that both $\alpha(t_1 - t_2)$ and $\beta(t_1 - t_2)$ are integers, which is not possible unless $t_1 = t_2$. It is not hard to show that the image $f(\mathbb{R})$ is an everywhere dense curve winding around the torus $S^1 \times S^1$. Therefore f is far from being an embedding, because the image of an embedding cannot be dense (see Proposition 4.5 below).

Note that the fact that $\mathbb{R}$ is not compact plays an essential role in these examples. Indeed, we have the following simple result.

Exercise 3.8. Show that if M is a compact manifold, then any injective immersion $M \longrightarrow N$ is an embedding.

Definition 3.9. Two smooth maps $f : M \longrightarrow N$ and $f' : M' \longrightarrow N'$ are called equivalent up to diffeomorphism if there exist diffeomorphisms $\phi : M \longrightarrow M'$ and $\psi : N \longrightarrow N'$ such that $\psi \circ f = f' \circ \phi$.

We will show in the next two theorems that any immersion is locally equivalent to a canonical embedding, and any submersion is locally equivalent to a canonical submersion.

Theorem 3.10 (**Local Immersion Theorem**). *Let M and N be manifolds of dimension n and m respectively. If $f : M \longrightarrow N$ is an immersion at $p \in M$, then there is a local representation of f at p which is the canonical embedding i.*

Proof. Let $g = \psi \circ f \circ \phi^{-1}$ be a local representation of f at p for a pair of coordinate systems (ϕ, ψ). We may suppose without loss of generality that $\phi(p) = 0$ and $\psi(f(p)) = 0$, and that the matrix of g at 0 is of the form

$$Jg(0) = \begin{pmatrix} A \\ B \end{pmatrix},$$

where A is a non-singular $n \times n$ matrix (the last condition may be realized by permuting the coordinates in ψ, if necessary). By changing the coordinates in $\mathbb{R}^m$

by a linear transformation $\mathbb{R}^m \longrightarrow \mathbb{R}^m$ whose matrix is

$$\begin{pmatrix} A^{-1} & O \\ -BA^{-1} & I_{m-n} \end{pmatrix},$$

where I_{m-n} is the identity matrix of order $m-n$ and O is a null matrix, the matrix $Jg(0)$ may be given the following form

$$\begin{pmatrix} A^{-1} & O \\ -BA^{-1} & I_{m-n} \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} I_n \\ O \end{pmatrix}.$$

Define a map $h : U \times \mathbb{R}^{m-n} \longrightarrow \mathbb{R}^m$, where U is the domain of g in $\mathbb{R}^n$, by

$$h(x, y) = g(x) + (0, y).$$

Then $g = h \circ i$, where $i : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the canonical embedding $x \mapsto (x, 0)$, and the matrix $Jh(0)$ is I_m. By the inverse function theorem, h is a local diffeomorphism at $0 \in \mathbb{R}^m$, and we have

$$\psi \circ f \circ \phi^{-1} = g = h \circ i \Rightarrow (h^{-1} \circ \psi) \circ f \circ \phi^{-1} = i.$$

Thus the local representation of f at p for the pair of coordinate systems $(\phi, h^{-1} \circ \psi)$ is the canonical embedding i. □

The following exercise points out that locally there is no distinction between immersion and embedding.

Exercise 3.11. Show that if $f : M \longrightarrow N$ is an immersion, then each point $p \in M$ has an open neighbourhood U such that $f|U$ is an embedding.

Theorem 3.12 (Local Submersion Theorem). *Let M and N be manifolds of dimension n and m respectively. If $f : M \longrightarrow N$ is a submersion at $p \in M$, then there is a local representation of f at p which is the canonical submersion s.*

Proof. As before, suppose that $g = \psi \circ f \circ \phi^{-1}$ be a local representation of f at p for a pair of coordinate systems (ϕ, ψ) such that $\phi(p) = 0$, $\psi(f(p)) = 0$, and that the Jacobian matrix of g at 0 is

$$Jg(0) = \begin{pmatrix} I_m & O \end{pmatrix},$$

after a linear change of coordinates in $\mathbb{R}^n$. Then, the map $h : U \longrightarrow \mathbb{R}^n$ given by $h(x) = (g(x), x_{m+1}, \ldots, x_n)$ has the Jacobian matrix I_n at $x = 0$, and we have $g = s \circ h$. Therefore $\psi \circ f \circ (h \circ \phi)^{-1}$ is the canonical submersion s. □

Exercises 3.13. (a) Show that any submersion is an open map (i.e. maps an open set onto an open set).

(b) Show that if M is compact and N is connected, then any submersion $f : M \longrightarrow N$ is surjective.

(c) Show that there is no submersion of a compact manifold into an Euclidean space.

Proposition 3.14. *Let M, N, and P be manifolds, and $f : M \longrightarrow N$ be a surjective submersion. Then a map $g : N \longrightarrow P$ is smooth if and only if the composition $g \circ f : M \longrightarrow P$ is smooth.*

Proof. If g is smooth, then $g \circ f$ is smooth by composition. To prove the converse, note that g is necessarily continuous, and, since the problem is local, we may suppose that f is the projection $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_m)$ from $\mathbb{R}^n$ onto $\mathbb{R}^m$, where $n = \dim M, m = \dim N$, and $n \geq m$. Then, by hypothesis, the map $g \circ f : (x_1, \dots, x_n) \mapsto g(x_1, \dots, x_m)$ is smooth. Therefore the map $g : (x_1, \dots, x_m) \mapsto g(x_1, \dots, x_m)$ is smooth. This means that g is smooth on $f(M)$, and hence on N, since f is surjective. □

Exercise 3.15. Show that if f and g are as in this proposition, then g is a submersion if and only if their composition $g \circ f$ is a submersion.

Exercises 3.16. (a) Show that if $f : M \longrightarrow N$ is a surjective submersion, then for each $x \in M$ there exist an open neighbourhood U of $f(x)$ in N, and a smooth map $g : U \longrightarrow M$ such that $f \circ g$ is the identity map on U.

The map g is called a local section of f.

(b) Suppose that $f : M \longrightarrow N$ is a smooth map such that every point of M is in the image of a smooth local section of f. Show that f is a submersion.

Exercise 3.17. If $f : M \longrightarrow N$ is a map and $y \in N$, then $f^{-1}(y)$ is called the fibre of f over y. Suppose that f is a surjective submersion. Show that if $g : M \longrightarrow P$ is a smooth map that is constant on the fibres of f, then there is a unique smooth map $h : N \longrightarrow P$ such that $h \circ f = g$.

Exercise 3.18. Show that a smooth map $f : M \longrightarrow N$ is a diffeomorphism if and only if it is bijective and a submersion.

Exercise 3.19. Let M, N, and P be manifolds, and $f : M \longrightarrow N$ be an immersion. Then show that a continuous map $g : P \longrightarrow M$ is smooth if and only if their composition $f \circ g : P \longrightarrow N$ is smooth.

Exercise 3.20. Prove the implicit function theorem in the following form. If $f : U \longrightarrow \mathbb{R}$, U open in $\mathbb{R}^n$, is a smooth map with $f(p) = q$ and $\partial f / \partial u_i(p) \neq 0$ for some i, then there is a smooth function

$$u_i = g(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$$

whose graph in some open neighbourhood of p in U is the set of solutions of the equation $f(u) = q$.

4. Submanifolds

Definition 4.1. Let N be an m-manifold. Then a subset M of N is called a **submanifold** of dimension n if for each point $p \in M$ there is a coordinate chart (U, ϕ) at p in N such that ϕ maps $M \cap U$ homeomorphically onto an open subset of $\mathbb{R}^n \subset \mathbb{R}^m$, where $\mathbb{R}^n$ is considered as the subspace of the first n coordinates in $\mathbb{R}^m$

$$\mathbb{R}^n = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_{n+1} = \dots = x_m = 0\}.$$

Then the collection

$$\{(M \cap U, \phi | M \cap U) \mid (U, \phi) \text{ is a chart in } N, M \cap U \neq \emptyset\}$$

is a smooth atlas of M.

Exercise 4.2. Show that a submanifold M of a manifold N is a second countable Hausdorff space.

Lemma 4.3. *Let M and N be manifolds of dimension n and m respectively. If M is a submanifold of N, then for each point $p \in M$ there is an open neighbourhood U of p in N and a submersion $g : U \longrightarrow \mathbb{R}^{m-n}$ such that $g^{-1}(0) = M \cap U$.*

Proof. By the above definition, there is a coordinate chart $\phi : U \longrightarrow \mathbb{R}^m$ about p in N such that if $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n}$, then $\phi^{-1}(\mathbb{R}^n \times \{0\}) = M \cap U$. Then $g = \pi \circ \phi$, where $\pi : \mathbb{R}^m \longrightarrow \mathbb{R}^{m-n}$ is the projection onto the second factor, is a submersion with $g^{-1}(0) = M \cap U$. □

Proposition 4.4. *A subset A of an m-manifold N is a submanifold if and only if A is the image of a smooth embedding $f : M \longrightarrow N$, where M is an n-manifold and $n \leq m$.*

Proof. If A is a submanifold of N, then it follows from the natural smooth structure on A derived from that of N that the inclusion of A in N is a smooth embedding. Conversely, suppose $f : M \longrightarrow N$ is a smooth embedding and $A = f(M)$. Then, by the local immersion theorem, for each $p \in M$ there exist a coordinate system $y_1, \ldots, y_m$ in an open neighbourhood V of $f(p)$ in N such that $A \cap V = \{q \in V | y_{n+1}(q) = \cdots = y_m(q) = 0\}$, and the restrictions of the remaining coordinate functions $y_1, \ldots, y_n$ to $A \cap V$ form a local chart on A at $f(p)$. Therefore A is a submanifold of N. □

Proposition 4.5. *If M is an n-dimensional submanifold of an m-manifold N where $n < m$, then M is not a dense subset of N.*

Proof. There is a coordinate chart (V, ψ) of N such that $U = M \cap V$ is non-empty, and $\psi(U) \subset \mathbb{R}^n \times \{0\}$. Then the non-empty open set $\psi^{-1}(\mathbb{R}^n \times (\mathbb{R}^{m-n} - \{0\}))$ of N lies in V and does not intersect U. So M cannot be dense in N. □

Exercises 4.6. Let M, N, and P, denote manifolds. Then show that

(1) if $f : N \longrightarrow P$ is a smooth map, then the restriction $f|M$ is also smooth; moreover, if f is an immersion, then $f|M$ is also an immersion.

(2) if M is a subset of N such that the inclusion $M \hookrightarrow N$ is an immersion, and $f : P \longrightarrow N$ is a smooth map with $f(P) \subset M$, then the map $f : P \longrightarrow M$ obtained by restricting the range of f may not be continuous. However, if

$$f : P \longrightarrow M$$

is continuous, then it is also smooth.

Definition 4.7. Let $f : M \longrightarrow N$ be a smooth map. Then a point $p \in M$ is called a **critical point** of f if f is not a submersion at p. Other points of M are called **regular points** of f. A point $q \in N$ is called a **critical value** of f if $f^{-1}(q)$ contains at least one critical point. Other points of N (including those for which $f^{-1}(q)$ is empty) are called **regular values** of f.

Theorem 4.8 (Preimage theorem). . *Let M and N be manifolds of dimension n and m respectively, where $n \geq m$. If q is a regular value of a smooth map $f : M \longrightarrow N$, then $f^{-1}(q)$ is a submanifold of M of dimension $n - m$.*

Proof. Since f is a submersion at a point $p \in f^{-1}(q)$, we can choose local coordinate systems about p and q such that $f(x_1, \ldots, x_n) = (x_1, \ldots, x_m)$, and q corresponds to $(0, \ldots, 0)$. Therefore, if U is the coordinate neighbourhood at p on which the functions $x_1, \ldots, x_n$ are defined, then $f^{-1}(q) \cap U$ is the set of points $(0, \ldots, 0, x_{m+1}, \ldots, x_n)$. Thus the functions $x_{m+1}, \ldots, x_n$ form a coordinate system on the relative open set $f^{-1}(q) \cap U$ of $f^{-1}(q)$. □

We may apply the theorem in the following situation. Let $m > n$, and N be an m-manifold. Let $f : N \longrightarrow \mathbb{R}^{m-n}$ be a smooth map. Then $M = f^{-1}(0)$ is the solution set of the system of equations

$$f_1(x_1, \ldots, x_n) = 0, \ldots, \ldots, f_{m-n}(x_1, \ldots, x_n) = 0,$$

where $f_i : N \longrightarrow \mathbb{R}$ are the components of f.

Proposition 4.9. *If f, N, and M are as above and rank $f = m - n$ at each point of N, then M is an n-dimensional submanifold of N.*

Proof. The proof follows immediately from the previous theorem. □

The converse is true locally.

Proposition 4.10. *Every n-submanifold M of an m-manifold N is locally definable as the set of common zeros of a set of functions $f_1, \ldots, f_{m-n} : U \longrightarrow \mathbb{R}$ such that*

$$\text{rank}\left(\frac{\partial f_i}{\partial x_j}\right) = m - n,$$

where U is a coordinate neighbourhood in N of a point in M with coordinates $x_1, \ldots, x_m$.

Proof. The proof follows immediately from the local immersion theorem. If $p \in M$, then there exists local coordinate system $x_1, \ldots, x_m$ defined on a neighbourhood U of p in N such that $M \cap U$ is given by the equations

$$x_{n+1} = 0, \ldots, x_m = 0.$$

□

5. Tangent spaces and derivative maps

Let U be an open set of a manifold M, and $C^\infty(U)$ denote the set of all smooth functions from U to $\mathbb{R}$. Let $p \in M$, and $\widetilde{C}^\infty(p)$ be the union of all $C^\infty(U)$ as U runs over all open neighbourhoods of p. This is an algebra over $\mathbb{R}$, because if $f \in C^\infty(U)$, and $g \in C^\infty(V)$, then $f + g, fg \in C^\infty(U \cap V)$, and $\lambda f \in C^\infty(U)$ for all $\lambda \in \mathbb{R}$. Two functions f and g as above are said to be **equivalent** (or have the same **germ** at p) if $f = g$ in a neighbourhood of p. The quotient set $C^\infty(p)$ of $\widetilde{C}^\infty(p)$ under this equivalence relation is also an algebra, called the **algebra of germs of smooth functions** at p.

In fact, $C^\infty(p)$ is the quotient algebra $\widetilde{C}^\infty(p)/\widetilde{C}^\infty_0(p)$, where $\widetilde{C}^\infty_0(p)$ is the ideal consisting of functions which vanish in a neighbourhood of p (neighbourhood depending on the function).

Definition 5.1. A **tangent vector** of M at a point $p \in M$ is the geometric name of what is called a derivation of the algebra $C^\infty(p)$ on $\mathbb{R}$. It is a linear functional $X_p : C^\infty(p) \longrightarrow \mathbb{R}$ satisfying the **Leibniz formula**

$$X_p(fg) = f(p) \cdot X_p(g) + g(p) \cdot X_p(f), \quad f, g \in C^\infty(p).$$

The formula implies that if f is a constant function, then $X_p f = 0$ for all $p \in M$.

The set $\tau(M)_p$ of all tangent vectors of M at p is called the **tangent space** of M at p, or the space of derivations at p. It is a vector space over $\mathbb{R}$, where the vector space operations are defined by $(X_p + Y_p)(f) = X_p(f) + Y_p(f)$, and $(\lambda X_p)(f) = \lambda X_p(f)$ for $X_p, Y_p \in \tau(M)_p$, $f \in C^\infty(p)$, and $\lambda \in \mathbb{R}$.

The geometric picture behind the definition will be clear after we prove that the dimension of the vector space $\tau(M)_p$ is n, which is also equal to the dimension of M.

Proposition 5.2. *If $\phi = (x_1, \ldots, x_n)$ is a coordinate system in M at p, then the operators*

$$\left[\frac{\partial}{\partial x_i}\right]_p : C^\infty(p) \longrightarrow \mathbb{R}, \quad i = 1, \ldots, n,$$

defined by $f \mapsto (\partial f/\partial x_i)(p)$ are tangent vectors of M at p, and they form a basis of the vector space $\tau(M)_p$.

Here $(\partial f/\partial x_i)(p)$ is the partial derivative as defined in §3, p. 8.

We first prove a lemma.

Lemma 5.3. [1] *Let $a \in \mathbb{R}^n$ and $f \in C^\infty(a)$. Then there exist functions $g_1, \ldots, g_n \in C^\infty(a)$ and a neighbourhood U of a in $\mathbb{R}^n$ contained in the intersection of the domains of $f, g_1, \ldots, g_n$ such that $g_i(a) = (\partial f/\partial u_i)(a), 1 \leq i \leq n$, and*

$$f(u) = f(a) + \sum_{i=1}^{n} (u_i - u_i(a)) \cdot g_i(u), u \in U,$$

where $u = (u_1, \ldots, u_n)$, $u_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, is the coordinate system in $\mathbb{R}^n$.

Proof. Define

$$g_i(u) = \int_0^1 \frac{\partial f}{\partial u_i}(t(u-a)+a)dt.$$

This is C^∞ in a neighbourhood of a, and $g_i(a) = (\partial f/\partial u_i)(a)$. Therefore

$$\begin{aligned} f(u) - f(a) &= \int_0^1 \frac{d}{dt} f(t(u-a)+a)dt \\ &= \int_0^1 \left\{ \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}(t(u-a)+a) \cdot (u_i - u_i(a)) \right\} dt \\ &= \sum_{i=1}^{n} g_i(u) \cdot (u_i - u_i(a)). \end{aligned}$$

[1] The lemma is not true for C^r manifolds where $r < \infty$, because the functions g_i may not be always C^r. In this case the space of derivations at p is infinite dimensional, and the tangent space is defined to be the space spanned by $[\partial/\partial x_i]_p$, see W.F. Newns and A.G. Walker, Tangent planes to a differentiable manifold, J. London Math. Soc. 31 (1956), 400-407.

□

Exercise 5.4. Show that if $a \in \mathbb{R}^n$ and $f \in C^\infty(a)$, then there exist functions $\lambda_{ij} \in C^\infty(a)$, and a neighbourhood U of a on which f and λ_{ij} are defined such that $\lambda_{ij}(a) = (\partial^2 f/\partial u_i \partial u_j)(a)$, and if $u \in U$ then

$$f(u) = f(a) + \sum_{i=1}^{n}(u_i - u_i(a)) \cdot \frac{\partial f}{\partial u_i}(a) + \sum_{i,j=1}^{n}(u_i - u_i(a))(u_j - u_j(a)) \cdot \lambda_{ij}(u).$$

Proof of the Proposition. That the operators $[\partial/\partial x_i]_p$ are tangent vectors is immediate from the definition. Next, for any $f \in C^\infty(p)$, use the lemma to write $f \circ \phi^{-1}$ in a neighbourhood of $a = \phi(p)$ as

$$f \circ \phi^{-1}(u) = f \circ \phi^{-1}(a) + \sum_{i=1}^{n}(u_i - u_i(a)) \cdot g_i(u),$$

where g_i is a C^∞ function in a neighbourhood of a with

$$g_i(a) = (\partial(f \circ \phi^{-1})/\partial u_i)(a).$$

Transferring this relation to a neighbourhood of p in M, write

$$f(x) = f(p) + \sum_{i=1}^{n}(x_i - x_i(p)) \cdot h_i(x),$$

where $h_i = g_i \circ \phi \in C^\infty(p)$ with $h_i(p) = g_i(a) = (\partial f/\partial x_i)(p)$. Now apply a derivation $X_p \in \tau(M)_p$ to f using the Leibniz formula :

$$X_p(f) = \sum_{i=1}^{n} X_p(x_i) \cdot h_i(p) = \sum_{i=1}^{n} X_p(x_i) \cdot \left[\frac{\partial f}{\partial x_i}\right]_p$$

(recall that $X_p(c) = 0$ for any constant function c). Thus in terms of the coordinate system $(x_1, \ldots, x_n)$, X_p takes the form

$$X_p = \sum_{i=1}^{n} X_p(x_i) \cdot \left[\frac{\partial}{\partial x_i}\right]_p .$$

Now $[\partial/\partial x_i]_p(x_j) = (\partial(u_j \circ \phi \circ \phi^{-1})/\partial u_i)(a) = \delta_{ij}$ (Kronecker delta). Therefore the vectors $\{[\partial/\partial x_i]_p\}$ are linearly independent, as may be seen by evaluating a linear combination of these vectors on x_j in turn. □

It follows that if U is an open neighbourhood of p, then $\tau(U)_p = \tau(M)_p$, because the definition of $\tau(M)_p$ uses only $C^\infty(p)$, and not the entire M. Also, the tangent space $\tau(M)_p$ is isomorphic to $\mathbb{R}^n$, where $[\partial/\partial x_i]_p$ corresponds to the i-th unit coordinate vector of $\mathbb{R}^n$, and therefore the tangent space $\tau(\mathbb{R}^n)_p$ can be identified with the set of all pairs (p, v), where $v \in \mathbb{R}^n$.

A **smooth curve** in M is a smooth map $\sigma : I \longrightarrow M$, where I is an open interval in $\mathbb{R}$. For each $t_0 \in I$, σ gives rise to a tangent vector $\dot\sigma(t_0) : C^\infty(p) \longrightarrow \mathbb{R}$ of M at $p = \sigma(t_0)$ defined by

$$\dot\sigma(t_0)(f) = \left[\frac{d}{dt} f(\sigma(t))\right]_{t=t_0},$$

which is the derivative of f along σ at p. The components of $\sigma(t)$ with respect to a local coordinate system $(x_1, \ldots, x_n)$ at p are the real-valued functions $\sigma_i(t) =$

$x_i(\sigma(t))$, and their derivatives $\dot{\sigma}_i(t_0) = (d(\sigma_i(t))/dt)(t_0)$ are the components of the tangent vector $\dot{\sigma}(t_0)$ with respect to the basis $[\partial/\partial x_i]_p$. Because, if (U, ϕ) is the coordinate chart at p for which $\phi = (x_1, \ldots, x_n)$, $x_i = u_i \circ \phi$, then $\sigma_i(t) = x_i(\sigma(t)) = u_i \circ \phi(\sigma(t))$, and therefore, by chain rule

$$\begin{aligned}\dot{\sigma}(t_0)(f) &= \frac{d}{dt}\Big[(f \circ \phi^{-1}) \circ (\phi \circ \sigma)\Big](t_0) \\ &= \sum_i \frac{\partial(f \circ \phi^{-1})}{\partial u_i}(\phi \circ \sigma(t_0)) \cdot \frac{\partial(u_i \circ \phi \circ \sigma)}{\partial t}(t_0) \\ &= \sum_i \frac{\partial f}{\partial x_i}(p) \cdot \frac{dx_i}{dt}(t_0).\end{aligned}$$

We also say that $\dot{\sigma}(t_0)$ is the **tangent vector** or **velocity vector** of σ at $\sigma(t_0)$. In the case when $M = \mathbb{R}^n$, this vector may be viewed as a line segment from $\sigma(t_0)$ to $\sigma(t_0) + \dot{\sigma}(t_0)$. Conversely, any tangent vector to M at p is associated to a smooth curve in this way. For, if $\phi = (x_1, \ldots, x_n)$ is a coordinate system at p, then a vector $\sum_i v_i[\partial/\partial x_i]_p \in \tau(M)_p$ is clearly tangent at p to the curve

$$t \mapsto \phi^{-1}(x_1(p) + tv_1, \ldots, x_n(p) + tv_n).$$

We may therefore define a tangent vector to M at p alternatively as follows. Consider the set of all smooth curves $\sigma : I \longrightarrow M$, where I is an open interval containing 0, such that $\sigma(0) = p$. Define an equivalence relation in this set by taking two curves σ and τ to be equivalent if $\dot{\sigma}(0) = \dot{\tau}(0)$. Then a tangent vector to M at p is an equivalence class of curves.

Exercise 5.5. Check that the relation on the set of all smooth curves as defined above is indeed an equivalence relation.

Example 5.6. Let $\langle\ ,\ \rangle$ denote the standard inner product in $\mathbb{R}^{n+1}$. Then the n-sphere S^n in $\mathbb{R}^{n+1}$ is given by

$$S^n = \{v \in \mathbb{R}^{n+1} | \langle v, v\rangle = 1\}.$$

Consider a smooth curve $\sigma : I \longrightarrow \mathbb{R}^{n+1}$ so that $\sigma(s) \in S^n$ for all $s \in I$, and $\sigma(0) = p$. Then $\langle \sigma(s), \sigma(s)\rangle = 1$. Differentiating this relation with respect to s at $s = 0$, we get

$$\langle \dot{\sigma}(0), \sigma(0)\rangle + \langle \sigma(0), \dot{\sigma}(0)\rangle = 0, \text{ or } \langle \dot{\sigma}(0), \sigma(0)\rangle = 0.$$

Since $\dot{\sigma}(0)$ is a vector in $\tau(S^n)_p$, the above relation says that $\tau(S^n)_p$ is the hyperplane in $\mathbb{R}^{n+1}$ orthogonal to $\sigma(0) = p$.

Definition 5.7. If $f : M \longrightarrow N$ is a smooth map between manifolds, then the **derivative map** or **differential** of f at a point $p \in M$ is a linear map $df_p : \tau(M)_p \longrightarrow \tau(N)_{f(p)}$ defined by

$$df_p(X_p)(g) = X_p(g \circ f),\ X_p \in \tau(M)_p,\ g \in C^\infty(f(p)).$$

Taking X_p as the velocity vector $\dot{\sigma}(0)$ of a smooth curve σ in M at $\sigma(0) = p$ with parameter t, the definition may be given in the following alternative form:

$$df_p(\dot{\sigma}(0))(g) = \frac{d}{dt}(g \circ f(\sigma(t)))(0).$$

We may rephrase the previous definition of the velocity vector $\dot{\sigma}(0)$ as follows

$$\dot{\sigma}(0) = d\sigma_0\Big(\frac{d}{dt}\Big),$$

where $d\sigma_0 : \tau(I)_0 = \mathbb{R} \longrightarrow \tau(M)_p$ is the derivative map of σ at 0, and d/dt is the basis of $\mathbb{R}$. Because

$$d\sigma_0\Big(\frac{d}{dt}\Big)(g) = \frac{d}{dt} g(\sigma(t))(0) = \dot{\sigma}(0)(g).$$

Let (U, ϕ) with $\phi = (x_1, \ldots, x_n)$, $x_i = u_i \circ \phi$, be a coordinate chart at p, and (V, ψ) with $\psi = (y_1, \ldots, y_m)$, $y_j = v_j \circ \psi$, be a coordinate chart at $q = f(p)$, where u_i (resp. v_j) are the coordinate functions on $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). Then

$$\begin{aligned} df_p\left(\left[\frac{\partial}{\partial x_i}\right]_p\right)(g) &= \frac{\partial}{\partial x_i}(g \circ f)(p) = \frac{\partial}{\partial u_i}(g \circ f \circ \phi^{-1})(\phi(p)) \\ &= \frac{\partial}{\partial u_i}(g \circ \psi^{-1} \circ \psi \circ f \circ \phi^{-1})(\phi(p)) = \frac{\partial}{\partial u_i}(\overline{g} \circ \overline{f})(\phi(p)), \end{aligned}$$

where $\overline{f} = \psi \circ f \circ \phi^{-1} : \phi(U) \longrightarrow \psi(V)$, and $\overline{g} = g \circ \psi^{-1} : \psi(V) \longrightarrow \mathbb{R}$ are smooth maps. By the chain rule, the last expression is equal to

$$\sum_{j=1}^{m} \frac{\partial \overline{f}_j}{\partial u_i}(\phi(p)) \cdot \frac{\partial \overline{g}}{\partial v_j}(\psi(q)) = \sum_{j=1}^{m} \frac{\partial f_j}{\partial x_i}(p) \cdot \frac{\partial g}{\partial y_j}(q).$$

Therefore

$$df_p\left(\left[\frac{\partial}{\partial x_i}\right]_p\right) = \frac{\partial f_1}{\partial x_i}(p)\left[\frac{\partial}{\partial y_1}\right]_{f(p)} + \cdots + \frac{\partial f_m}{\partial x_i}(p)\left[\frac{\partial}{\partial y_m}\right]_{f(p)}.$$

Therefore the i-th column vector of the matrix of the linear map df_p with respect to the bases $[\partial/\partial x_i]_p$ and $[\partial/\partial y_j]_{f(p)}$ of the tangent spaces $\tau(M)_p$ and $\tau(N)_{f(p)}$ is

$$\Big(\frac{\partial f_1}{\partial x_i}(p), \ldots, \frac{\partial f_m}{\partial x_i}(p)\Big).$$

Therefore the matrix of df_p is the Jacobian matrix of f at p, as defined in §(1.4)

$$(Jf)(p) = \left(\frac{\partial f_i}{\partial x_j}(p)\right).$$

Thus if we represent a tangent vector $X_p = \sum_i a_i\Big(\frac{\partial}{\partial x_i}\Big)_p$ by the $n \times 1$ matrix $A = (a_i)$, then the tangent vector $df_p(X_p)$ is represented by the $m \times 1$ matrix $(Jf)(p) \cdot A$. In particular, for the coordinate chart (U, ϕ),

$$d\phi_p\Big(\sum_i a_i\Big(\frac{\partial}{\partial x_i}\Big)_p\Big) = (a_1, \ldots, a_n),$$

If $f : M \longrightarrow N$ and $g : N \longrightarrow L$ are smooth maps of manifolds, then

$$d(g \circ f)_p = dg_{f(p)} \circ df_p, \quad p \in M.$$

For. if $X_p \in \tau(M)_p$ and $h \in C^\infty(g(f(p)))$, then

$$d(g \circ f)_p(X_p)(h) = X_p(h \circ g \circ f) = df_p(X_p)(h \circ g) = dg_{f(p)}(df_p(X_p))(h).$$

In terms of local coordinates this computation exhibits the chain rule and the multiplicative behaviour of Jacobian matrices.

6. Tangent Bundles and Vector Fields

Definition 6.1. The **tangent bundle** $\tau(M)$ of M is the disjoint union of all tangent spaces $\tau(M)_p$ as p runs over M.

This is the set of all ordered pairs (p, v) such that $v \in \tau(M)_p$. The map $\pi : \tau(M) \longrightarrow M$, given by $(p, v) \mapsto p$, is called the **projection map** of the tangent bundle. The following theorem shows we can pull back the differential structure on M by π to obtain a unique differential structure on $\tau(M)$.

Theorem 6.2. *If M is a manifold of dimension n, then its tangent bundle $\tau(M)$ is a manifold of dimension $2n$.*

Proof. Each chart (U, ϕ) of M determines a map $\tau_\phi : \pi^{-1}(U) \longrightarrow \phi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$ given by $\tau_\phi(p, v) = (\phi(p), d\phi_p(v))$. Clearly, τ_ϕ is a bijection with inverse τ_ϕ^{-1} given by $\tau_\phi^{-1}(a, w) = (p, d\phi_p^{-1}(w))$ where $p = \phi^{-1}(a)$. For two compatible charts (U, ϕ) and (V, ψ) of M, the map $\tau_\psi \circ \tau_\phi^{-1} : \phi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n$ is given by

$$\begin{aligned} \tau_\psi \circ \tau_\phi^{-1}(a, w) &= \tau_\psi(p, d\phi_p^{-1}(w)) \\ &= (\psi(p), d\psi_p \circ d\phi_p^{-1}(w)) \\ &= (\psi \circ \phi^{-1}(a), d\psi_p \circ d\phi_p^{-1}(w)), \end{aligned}$$

where $p = \phi^{-1}(a)$. Therefore $\tau_\psi \circ \tau_\phi^{-1}$ is a homeomorphism. It follows that $\tau(M)$ has a unique topology which makes each τ_ϕ a homeomorphism. Moreover, since $\tau_\psi \circ \tau_\phi^{-1}$ is a diffeomorphism, the family of charts $\{(\pi^{-1}(U), \tau_\phi)\}$ constitute a smooth atlas on $\tau(M)$. Thus $\tau(M)$ is a smooth manifold. □

Exercise 6.3. Complete the proof of the above theorem by showing that $\tau(M)$ is second countable and Hausdorff. Also show that the projection $\pi : \tau(M) \longrightarrow M$ is a smooth map.

Exercise 6.4. Show that a smooth map $f : M \longrightarrow N$ between manifolds induces a smooth map $df : \tau(M) \longrightarrow \tau(N)$ which is defined by $df(p, v) = (f(p), df_p(v))$.

Definition 6.5. A **vector field** X on M is a map $X : M \longrightarrow \tau(M)$ such that the value of X at $p \in M$ is a tangent vector $X_p \in \tau(M)_p$.

For any $f \in C^\infty(U)$, a vector field X defines a function $Xf : U \longrightarrow \mathbb{R}$ by $(Xf)(p) = X_p(f)$. A vector field X is called a **smooth vector field** if, for every $p \in M$, $f \in C^\infty(p)$ implies $Xf \in C^\infty(p)$ also.

Thus a smooth vector field X may be considered as a map

$$X : C^\infty(M) \longrightarrow C^\infty(M)$$

given by $f \mapsto Xf$. We have

(i) $X(\lambda f + \mu g) = \lambda Xf + \mu Yf$,

(ii) $X(fg) = f(Xg) + (Xf)g$,

for $f, g \in C^\infty(M)$, and $\lambda, \mu \in \mathbb{R}$.

Exercise 6.6. Show that a smooth vector field X on M is completely determined by its action on smooth functions on M satisfying the above properties (i) and (ii).

Exercise 6.7. Show that if f is a constant function, then $Xf = 0$.

The set of all smooth vector fields on M is denoted by $\mathfrak{X}(M)$. This is a module over the ring $C^\infty(M)$, where the module operations are given by

$$(X+Y)f = Xf + Yf, \text{ and } (fX)g = f(Xg),$$

for $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$.

If (U, ϕ) is a coordinate chart in M with $\phi = (x_1, \ldots, x_n)$, then for each $i = 1, \ldots, n$, the assignment $p \mapsto [\partial/\partial x_i]_p$ is a smooth vector field $\partial/\partial x_i$ on U. The tangent vectors $([\partial/\partial x_i]_p)$ are linearly independent at each point $p \in U$. Therefore, if X is a vector field on U, then X may be written as

$$X = \sum_{i=1}^{n} Xx_i \cdot \frac{\partial}{\partial x_i}.$$

The functions Xx_i are called the **components** of X.

Exercise 6.8. Show that a vector field X is smooth if and only if its components Xx_i are smooth for every coordinate system ϕ.

Lemma 6.9. *If X is a smooth vector field on an open neighbourhood U in M, and $p \in U$, then there is an open neighbourhood V of p in U, and a smooth vector field $\widehat{X}$ on M which agrees with X on V.*

Proof. Let K be a closed neighbourhood of p in U, and let V be the interior of K. Then, by the Smooth Urysohn's Lemma (see Lemma 1.7 (Part 2)), there is a smooth function $\phi : M \longrightarrow \mathbb{R}$ with support in U such that $\phi = 1$ on K. Then define a vector field $\widehat{X}$ on M by

$$\begin{aligned} \widehat{X}(q) &= \phi(q)X(q) \text{ if } q \in U \\ &= 0 \text{ if } q \notin U \end{aligned}$$

Clearly this is the required vector field. □

7. Manifolds with boundary

We extend the notion of manifolds so as to include manifolds with boundary. For example, the disk $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ is a manifold with boundary which is the $(n-1)$-sphere

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}.$$

Let $\mathbb{R}^n_+$ and $\partial\mathbb{R}^n_+$ denote the subsets of $\mathbb{R}^n$ given by

$$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}, \quad \partial\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}.$$

We call $\mathbb{R}^n_+$ the half space of $\mathbb{R}^n$, and $\partial\mathbb{R}^n_+$ the boundary of $\mathbb{R}^n_+$ (a more general definition says that a half space in $\mathbb{R}^n$ is an affine hyperplane, but we will not consider this). Note that we may identify $\partial\mathbb{R}^n_+$ with $\mathbb{R}^{n-1} \subset \mathbb{R}^n$.

Lemma 7.1. *Any linear isomorphism $\mathbb{R}^n \longrightarrow \mathbb{R}^n$, which maps $\partial\mathbb{R}^n_+$ onto itself, maps $\mathbb{R}^n_+$ onto itself.*

Proof. The proof is obvious. Because, we may identify $\partial\mathbb{R}^n_+ \times \mathbb{R}$ with $\mathbb{R}^n$ by the linear isomorphism $\alpha(v_0, r) = v_0 + re_1$ (e_1 = unit vector along the first coordinate axis), so that $\mathbb{R}^n_+ = \alpha(\partial\mathbb{R}^n_+ \times \mathbb{R}_+)$. □

If U is an open subset in $\mathbb{R}^n_+$, then its boundary ∂U is the subset $\partial U = U \cap \partial\mathbb{R}^n_+$, and its interior $\mathrm{Int}(U)$ is the subset $\mathrm{Int}(U) = U - \partial U$. Thus $\mathrm{Int}(U)$ is open in $\mathbb{R}^n$, and ∂U is open in $\mathbb{R}^{n-1}$.

We may define smooth maps on open subsets of $\mathbb{R}^n_+$ by means of Definition 2.4. Thus a map $f : U \longrightarrow V$, where U is open in R^n_+ and V open in $\mathbb{R}^m_+$, is smooth if for each $x \in U$ there exist an open neighbourhood U_1 of x in $\mathbb{R}^n$, an open neighbourhood V_1 of $f(x)$ in $\mathbb{R}^m$, and a smooth map $f_1 : U_1 \longrightarrow V_1$ such that $f_1|U \cap U_1 = f|U \cap U_1$.

The notion of derivative of map also extends naturally. Consider a smooth map $f : U \longrightarrow \mathbb{R}^m$, where U is open in $\mathbb{R}^n_+$. Then, if $x \in \mathrm{Int}(U)$, we already know what is df_x. If $x \in \partial U$, then, since f is smooth at x, f extends to a smooth map F in an open neighbourhood of x in $\mathbb{R}^n$. In this case, we define df_x to be the derivative map dF_x, which is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. The definition is independent of the choice of the extension F, that is, if F' is another local extension of f, then $dF'_x = dF_x$. To see this, note that if V and V' are the domains of F and F' respectively, and if $\{x_j\}$ is a sequence of points in $V \cap V' \cap \mathrm{Int}(U)$ converging to x, then, since F and F' agree on $V \cap V' \cap \mathrm{Int}(U)$, we have $dF_{x_j} = dF'_{x_j}$, as sequences in the vector space $L(\mathbb{R}^n, \mathbb{R}^m)$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. This implies, as $j \to \infty$, that $dF_x = dF'_x$, because the derivative maps $dF,\ dF' : V \cap V' \longrightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ are continuous.

It follows that the definition of differentiability of $f : U \longrightarrow \mathbb{R}^m$ at a point $p \in U$ may be obtained from Definition 3.1, just by supposing U is an open subset of the half space $\mathbb{R}^n_+$ and keeping the other things the same. The derivative map $df_a : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ in the new situation will have the same properties (1)-(6) of Proposition 3.2.

Exercises 7.2. Show that

(1) if $f : U \longrightarrow \mathbb{R}^m$ is differentiable at $a \in U$, where U is open in $\mathbb{R}^n_+$, then

$$\begin{aligned} df_a(v) &= \lim_{t\to 0+} \frac{f(a+tv) - f(a)}{t} \text{ if } v \in \mathbb{R}^n_+ \\ &= \lim_{t\to 0-} \frac{f(a+tv) - f(a)}{t} \text{ if } -v \in \mathbb{R}^n_+. \end{aligned}$$

(2) if $f, g : U \longrightarrow \mathbb{R}^m$ are differentiable maps, where U is open in $\mathbb{R}^n$, such that f and g agree on $U \cap \mathbb{R}^n_+$, then $df_a = dg_a$ for $a \in U \cap \mathbb{R}^n_+$.

Lemma 7.3. *If $f : U \longrightarrow \mathbb{R}^m_+$ is differentiable, where U is open in $\mathbb{R}^n$, such that f maps $a \in U$ into $f(a) \in \partial\mathbb{R}^m_+$, then df_a maps $\mathbb{R}^n$ into $\partial\mathbb{R}^m_+$.*

Proof. Let $v \in \mathbb{R}^n$. Then

$$df_a(v) = \lim_{t\to 0} \frac{f(a+tv) - f(a)}{t}.$$

Therefore given an $\epsilon > 0$, there is a $\delta > 0$ such that if $a + tv \in U$, then

$$\left\| df_a(v) - \frac{f(a+tv) - f(a)}{t} \right\| < \epsilon$$

for all $t \in (-\delta, \delta)$, $t \neq 0$. Write

$$u_t = df_a(v) - \frac{f(a+tv) - f(a)}{t},$$

where t is as above. Then

$$t(df_a(v) - u_t) = f(a+tv) - f(a).$$

Let $F_1[v]$ denote the first coordinate of the vector v. Then,
since $-f(a) \in \partial\mathbb{R}^m_+ \subset \mathbb{R}^m_+$, and $f(a+tv) \in \mathbb{R}^m_+$, we have

$$t \cdot F_1[df_a(v) - u_t] = F_1[f(a+tv) - f(a)] \geq 0.$$

Therefore, if $0 < t < \delta$, then

$$F_1[df_a(v)] \geq F_1[u_t] > -\epsilon,$$

and if $-\delta < t < 0$, then

$$F_1[df_a(v)] \leq F_1[u_t] < \epsilon.$$

Therefore $-\epsilon < F_1[df_a(v)] < \epsilon$, and as $\epsilon \to 0$, we have $F_1[df_a(v)] = 0$. Therefore $df_a(v) \in \partial\mathbb{R}^m_+$. □

Theorem 7.4. (**Invariance of Interior and Boundary**). Let $f : U \longrightarrow V$ be a diffeomorphism, where U and V are open subsets of $\mathbb{R}^n_+$, then

(a) $x \notin \partial U \Leftrightarrow f(x) \notin \partial V$,

(b) $f \mid \text{Int}(U)$, and $f \mid \partial U$ are diffeomorphisms.

Proof. The derivative map $df_a : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isomorphism for each $a \in U$, by the functorial properties of derivative (Proposition 3.2). Therefore, by the preceding lemma, no interior point of U can be mapped onto a boundary point of V, and conversely. Thus f induces bijections $\text{Int}U \longrightarrow \text{Int}V$ and $\partial U \longrightarrow \partial V$. These are actually diffeomorphisms, because the restriction of f to any subset of U is always a smooth map.

□

Definition 7.5. A second countable Hausdorff space M is called a smooth n-**manifold with boundary** if is satisfies all the conditions of a smooth manifold, with the exception that now we allow coordinate neighbourhoods to map onto open subsets in $\mathbb{R}^n_+$.

If $\phi : U \longrightarrow V \subset \mathbb{R}^n_+$ is such a coordinate chart, where U is open in M and V is open in $\mathbb{R}^n_+$, then a point of $\phi^{-1}(\partial V)$ is called a boundary point for the chart (U, ϕ). The definition does not depend on the chart. For, if (U, ϕ) and (V, ψ) are two coordinate charts around $x \in M$ with $\phi(x) \in \partial\mathbb{R}^n_+$ and $\psi(x) \in \text{Int}(\mathbb{R}^n_+)$, then the diffeomorphism $\psi \circ \phi^{-1}$ will map a boundary point of $\mathbb{R}^n_+$ onto an interior point of $\mathbb{R}^n_+$. This is not possible by the invariance of boundary as described in Theorem 7.4. The collection of all boundary points is the **boundary** of M, which is denoted by ∂M.

Theorem 7.6. *The boundary ∂M of an n-manifold M is a manifold of dimension $n-1$, and ∂M has no boundary.*

Proof. We have already seen that if x is a boundary point with respect to one coordinate system, then it remains a boundary point relative to any other coordinate system. If $\phi : U \longrightarrow V \subset \mathbb{R}^n_+$ is a coordinate chart in M, then $\phi^{-1}(\partial V) = U \cap \partial M$ is an open set in ∂M, and $(U \cap \partial M, \lambda \circ \phi)$ is a coordinate chart for ∂M, where $\lambda : \partial R^n_+ \longrightarrow \mathbb{R}^{n-1}$ is a linear isomorphism. The collection of all such charts is a smooth atlas on ∂M. Thus the boundary ∂M is a manifold of dimension $n-1$. ☐

The interior of M is the set $\text{Int}M = M - \partial M$. It is a manifold of the same dimension as M, and it has no boundary.

Exercise 7.7. Show that if $f : M \longrightarrow N$ is a diffeomorphism, then $f(\partial M) = \partial N$ and $f(\text{Int}M) = \text{Int}N$.

The notion of submanifold can also be extended.

Definition 7.8. An m-submanifold N of an n-manifold M with boundary satisfies the same conditions as when M is without boundary, except that, for every coordinate chart (U, ϕ), $\phi : U \longrightarrow \mathbb{R}^n_+$, $\phi^{-1}(\mathbb{R}^m_+) = U \cap N$, where $\mathbb{R}^m_+$ is the subspace of the first m coordinates in $\mathbb{R}^n_+$.

A map on a manifold with boundary is smooth, if it is locally extendable to a smooth map. The concepts of rank, immersion, submersion, embedding, and diffeomorphism remain exactly the same as before. However, there are two kinds of submanifolds N of M arising from two kinds of embeddings, namely, embeddings of a manifold into a manifold without boundary, or embeddings of a manifold into a manifold with boundary. Consider, for example, a closed interval I embedded in $\mathbb{R}^n_+$; I may lie entirely in $\text{Int}(\mathbb{R}^n_+)$, or I may have a boundary point in $\partial\mathbb{R}^n_+$. The two cases are essentially distinct, although Proposition 4.4 holds for each of them. For example, given two submanifolds of $\mathbb{R}^n_+$ of the first kind, there exists a diffeomorphism of $\mathbb{R}^n_+$ carrying one to the other, but there cannot exist a diffeomorphism of $\mathbb{R}^n_+$ carrying a submanifold of the first kind into one of the second kind (why?).

In general, there is no relation between ∂N and ∂M, when N is a submanifold of M. We define a special kind of submanifold N whose boundary is nicely placed in the ambient manifold M.

Definition 7.9. An m-submanifold N of an n-manifold M is a **neat submanifold** of M if N is a closed subset of M, and

(a) each point $p \in N$ has a chart (U, ϕ) at p in M, where $\phi : U \longrightarrow \mathbb{R}^n_+$, such that $\phi^{-1}(\mathbb{R}^m_+) = U \cap N$,
(b) each point $p \in \partial N$ has a chart (U, ϕ) at p in M, where $\phi : U \longrightarrow \mathbb{R}^n_+$, such that $\phi^{-1}(\partial\mathbb{R}^m_+) = U \cap \partial N$,

The definition implies that N meets ∂M in the same way as $\mathbb{R}^m_+$ meets $\partial\mathbb{R}^n_+$. Indeed, $\partial\mathbb{R}^m_+ = \mathbb{R}^m_+ \cap \partial\mathbb{R}^n_+$ implies $\partial N = N \cap \partial M$. In particular, if $\partial N = \emptyset$, then N is disjoint from ∂M, and so N is a submanifold of $\text{Int}M$. Note that a curve with end points in a manifold with boundary is not a neat submanifold of M unless its end points lie in ∂M.

Exercise 7.10. Show that a closed subset A of an n-manifold M is a neat submanifold of dimension m if and only if at each point $p \in A$ there is a chart (U, ϕ) in

M and a submersion $f : U \longrightarrow \mathbb{R}^{n-m}$ such that f is also a submersion on $U \cap \partial M$, and $f^{-1}(0) = U \cap A$.

Exercise 7.11. Extend Definition 4.7 of regular value of a smooth map

$$f : M \longrightarrow N$$

as follows. A point $q \in N$ is a regular value of f if (1) f is a submersion at every point $p \in f^{-1}(q)$, and (2) $f \mid \partial M$ is a submersion at every point $p \in f^{-1}(q) \cap \partial M$. If $p \in \text{Int}M$, then the condition (2) does not arise, and if $p \in \partial M$, then condition (1) is redundant, as it follows from the condition (2).

Show that if q is a regular value of f, then $f^{-1}(q)$ is a neat submanifold of M.

It may be noted that the definitions of tangent vector and tangent bundle remain the same in the context of manifold with boundary.