# SIMPLE AND SEMISIMPLE FINITE DIMENSIONAL ALGEBRAS

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Let $A$ be a ring (with identity). We call $A$ *simple* if it is non-zero (that is, $1 \neq 0$) and has no proper non-zero two-sided ideals. We call $A$ *semisimple* if it is semisimple as a (left) module over itself.[1] Our goal here is to prove structure theorems for simple and semisimple finite dimensional algebras over a field.

## 1. Preliminaries

- Simple module and Schur's lemma
- every simple module is a quotient of the ring
- Semisimple modules: equivalent conditions
- Isotypical components
- semisimplicity of group ring
- modules for the group ring are representations of the group

## 2. Simple finite dimensional algebras [s:s]

Let $F$ be a field and $A$ an (associative) algebra (with identity) over $F$ with $\dim_F A < \infty$.

[ss:example]

2.1. **An example.** The prototypical example of such an algebra $A$ is $\mathrm{End}_F V$, the ring of $F$-linear endomorphisms of a finite dimensional (non-zero) $F$-vector space $V$.

There is a natural bijective inclusion-reversing corresponding between subspaces of $V$ and left ideals of $\mathrm{End}_F V$: to a subspace $W$ is associated the left ideal $W^\perp$ consisting of all endomorphisms that vanish on $W$. There is also a natural inclusion-preserving correspondence between subspaces and right ideals: given a subspace $W$, the set $\mathfrak{r}_W$ of all endomorphisms with range contained in $W$ is a right ideal. In particular, there are only two left ideals that are also right ideals: $0^\perp$ which is the whole of $\mathrm{End}_F V$ and $V^\perp$ which is $0$.

The $F$-dimension of $W^\perp$ is $\dim V \cdot \dim(V/W)$, that of $\mathfrak{r}_W$ is $\dim V \cdot \dim W$. If $W_1$ and $W_2$ are subspaces of the same dimension, then $W_1^\perp$ and $W_2^\perp$ are isomorphic as $A$-modules: choosing $g$ to be a linear isomorphism of $V$ that maps $W_1$ to $W_2$, we have $\phi \mapsto \phi g$ an $A$-module isomorphism of $W_2^\perp$ onto $W_1^\perp$.

Any minimal (non-zero) left ideal $\ell$ of $A$ arises as $H^\perp$ for some hyperplane $H$ of $V$. It is isomorphic to $V$ as $A$-modules. Indeed, choosing a non-zero element of $V/H$ (two such

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[1] Logically, $A$ should be called "left semisimple" rather than just "semisimple". But, as we will shortly see, at least in the case of finite dimensional algebras, left semisimple and right semisimple are equivalent.

elements are scalar multiples of each other), any element of $H^\perp$ is determined by the value at this element, and the resulting evaluation map from $H^\perp$ to $V$ defines an $A$-isomorphism.

$\mathrm{End}_F V$ is semisimple: if we choose as many hyperplanes $H_1, \ldots, H_n$ in $V$ as the dimension $n$ of $V$ such that they intersect trivially, then $\mathrm{End}_F V$ is the (internal) direct sum of the corresponding minimal left ideals.

The algebra $\mathrm{End}_F V$ is (non-canonically) isomorphic to its opposite. To see this, fix a non-degenerate bilinear form on $V$. Using this form, we can identity $V$ with its dual $V^*$ and so also $\mathrm{End}_F V$ with $\mathrm{End}_F V^*$. The association $\varphi \leftrightarrow \varphi^*$, where $\varphi^*$ is the transpose of $\varphi$, defines an isomorphism between $(\mathrm{End}_F V)^{\mathrm{opp}}$ and $\mathrm{End}_F V^*$.

[ss:simple]

2.2. **The structure of a simple algebra.** Assume that $F$ is algebraically closed and let $\ell$ be a minimal (non-zero) left ideal of a simple finite dimensional algerba $A$ over $F$. The natural map $\rho : A \to \mathrm{End}_F \ell$ defining the action of $A$ on $\ell$ is injective since the kernel is a proper two-sided ideal and $A$ is simple. We claim that $\rho$ is surjective and therefore an isomorphism. Indeed this follows from the following famous result.

[l:burnside]

**Lemma 1.** (Burnside) *Let $B$ be an algebra (not necessarily finite dimensional) over an algebraically closed field $F$, and $V$ a simple finite dimensional $B$-module. Then the canonical map $B \to \mathrm{End}_F V$ defining the action of $B$ on $V$ is an epimorphism.*

PROOF: It is enough to show the following: given a $F$-basis $v_1, \ldots, v_n$ of $V$, and arbitrary elements $w_1, \ldots, w_n$ of $V$, there exists an element $b$ in $B$ such that $bv_1 = w_1, \ldots, bv_n = w_n$. Consider the $B$-module map $\psi : B \to \ell^{\oplus n}$ defined by $b \mapsto (bv_1, \ldots, bv_n)$. Since $V$ is simple, $V^{\oplus n}$ is semisimple. The image of $\psi$ being a submodule of $V^{\oplus n}$, it is isomorphic to $V^{\oplus r}$ with $r \leq n$. By Schur's lemma, any $B$-module map between $V^{\oplus r}$ and $V^{\oplus n}$ is given by $(u_1, \ldots, u_r) \mapsto (u_1, \ldots, u_r)M$, where $M$ is an $r \times n$ matrix over $\mathrm{End}_B V$, and we think of $V$ and its direct sums naturally as right modules over $\mathrm{End}_B V$. Since 1 in $B$ maps to $(v_1, \ldots, v_n)$ under $\psi$, it follows that there exists $(u_1, \ldots, u_r)$ in $V^{\oplus r}$ such that $(u_1, \ldots, u_r)M = (v_1, \ldots, v_n)$. Our hypothesis that $F$ is algebraically closed now means that $\mathrm{End}_B V$ is $F$. Since $v_1, \ldots, v_n$ are linearly independent, this is possible only if $r = n$, and the lemma is proved. $\square$

Structure of a simple algebra

We've thus proved the following theorem:

[t:simple]

**Theorem 2.** *A finite dimensional simple algebra over an algebraically closed field is isomorphic to the ring of linear endomorphisms of any of its minimal left ideals.*

Combining the theorem with the facts proved in §2.1, we conclude that such an algebra is semisimple: indeed, it is isomorphic to $\ell^{\oplus n}$ where $\ell$ is any minimal left ideal and $n$ the $F$-dimension of $\ell$. The $F$-dimension of such an algebra is the square of the dimension of any of its minimal left ideals.

[c:burnside]

**Corollary 3.** *Let $F$ be an algebraically closed field. Let $V$, $W$ be finite dimensional simple modules respectively for $F$-algebras $B$ and $C$. Then $V \otimes W$ is a simple $B \otimes C$-module. Moreover, every simple finite dimensional $B \otimes C$-module arises thus.*

Proof: Since the canonical maps from $B$ and $C$ respectively to $\mathrm{End}_F V$ and $\mathrm{End}_F W$ are surjective (by Lemma 1), the image of $B \otimes C$ under their tensor product is $\mathrm{End}_F V \otimes \mathrm{End}_F W = \mathrm{End}_F(V \otimes W)$, which proves the first assertion. For the converse, given a finite dimensional $B \otimes C$-module $X$, let $V$ be a simple $B$-submodule (this exists because $\dim_F X < \infty$). Now $\mathrm{Hom}_B(V, X)$ is naturally a $C$-module (by the action on $X$, since the actions of $B$ and $C$ commute), and we have an evaluation morphism $\mathfrak{ev} : V \otimes \mathrm{Hom}_B(V, X) \to X$, which is $B \otimes C$-linear. The image of $\mathfrak{ev}$ is non-zero, and therefore onto $X$ since $X$ is simple. Let $W$ be any simple $C$-submodule of $\mathrm{Hom}_B(V, X)$. Since $\mathfrak{ev}(V \otimes \varphi) \neq 0$ for $\varphi \neq 0$ (by the nature of $\mathrm{Hom}_B(V, X)$), we have $\mathfrak{ev}(V \otimes W)$ is non-zero and therefore all of $X$. In other words, $\mathfrak{ev} : V \otimes W \to X$ is an isomorphism (that it is injective uses the first part). □

[sss:nonac]

2.2.1. The case when $F$ is not algebraically closed. If the base field is not algebraically closed, then the result above still holds in a slightly modified form. Let $\ell$ be a minimal left ideal and put $\mathbb{D} := \mathrm{End}_A \ell$. By Schur's lemma, $\mathbb{D}$ is a division subring of $\mathrm{End}_F \ell$. We now consider $\ell$ as a (finite dimensional) vector space over $\mathbb{D}$: observe that $\dim_{\mathbb{D}} \ell \cdot \dim_F \mathbb{D} = \dim_F \ell$. The claim now is that $A \to \mathrm{End}_F V$ maps onto $\mathrm{End}_{\mathbb{D}} \ell$.

To prove the claim, choose $v_1, \ldots, v_n$ to be a $\mathbb{D}$-basis of $\ell$ and proceed as before. The matrix $M$ will now have entries over $\mathbb{D}$, and once again $r = n$ is forced by the linear independence of $v_1, \ldots, v_n$ (this time over $\mathbb{D}$). We've thus proved the following theorem:

Structure of a simple algebra: base field not necessarily algebraically closed

**Theorem 4.** *A finite dimensional simple algebra over a field is isomorphic to* $\mathrm{End}_{\mathbb{D}} \ell$, *where* $\ell$ *is any minimal left ideal and* $\mathbb{D}$ *the division ring* $\mathrm{End}_A \ell$.

[ss:exrevisit]

2.3. **Revisiting the prototypical example.** In the light of §2.2.1, it is natural to look back at Example 2.1 and generalize it. To this end, let $\mathbb{D}$ be a division ring with $F$ imbedded centrally in it and $\dim_{\mathbb{D}} F < \infty$. Let $V$ be a finite dimensional (left) $\mathbb{D}$-vector space, and put $A = \mathrm{End}_{\mathbb{D}} V$.

The correspondence of left ideals of $A$ with $\mathbb{D}$-subspaces of $V$ works in a similar fashion. So does the correspondence of right ideals. In particular, $A$ is simple as before. Subspaces of the same dimension correspond to isomorphic one-sided ideals. Any minimal left ideal of $A$ is isomorphic to $V$ as $A$-modules.

The dual $\mathrm{Hom}_{\mathbb{D}}(V, \mathbb{D})$ is naturally a right $\mathbb{D}$-vector space and so a left $\mathbb{D}^{\mathrm{opp}}$-vector space. The association $\varphi \leftrightarrow \varphi^*$, where $\varphi^*$ is the transpose of $\varphi$, defines an isomorphism of $(\mathrm{End}_{\mathbb{D}} V)^{\mathrm{opp}}$ with $\mathrm{End}_{\mathbb{D}^{\mathrm{opp}}} V^*$.

## 3. Semisimple finite dimensional algebras

[s:ss]

[ss:exampless]

3.1. **Prototypical examples of semisimple algebras.** Let $V_1, \ldots, V_k$ be finite dimensional vector spaces over the (arbitrary) field $F$ and $A$ be the product $\mathrm{End}_F V_1 \times \cdots \times \mathrm{End}_F V_n$. The vector spaces $V_j$ are naturally $A$-modules: indeed, each $V_j$ is an $\mathrm{End}_F V_j$-module and thus also an $A$-module via the projection $A \to \mathrm{End}_F V_j$. As observed in §2.1, each $\mathrm{End}_F V_j$ is semisimple as a module over itself. So each $\mathrm{End}_F V_j$ is a semisimple $A$-module and $A$ being the direct sum of these as a module is also semisimple.

[ss:ssimple]

3.2. **The structure of a semisimple algebra.** Our goal is to show that every finite dimensional semisimple algebra over an algebraically closed field is of the prototypical form described in §3.1. The following simple, beautiful observation is used crucially in the proof:

> *Let $R$ be a ring (with identity) and $L$ denote $R$ as a left module over itself. Then the ring $\mathrm{End}_R(L)$ of $R$-endomorphisms of $L$ is naturally isomorphic to the opposite ring $R^{\mathrm{opp}}$ of $R$.*

Indeed, $L$ being generated by 1, any $R$-endomorphism of $L$ is determined by where it maps 1, say to $r$, but then it must be right multiplication by $r$.

[t:wedderburn]

**Theorem 5.** (Wedderburn) *Let $A$ a finite dimensional semisimple algebra over an arbitrary field $F$. Let left ideals $\ell_1, \ldots, \ell_k$ be so chosen that no two of them are isomorphic and together they represent all isomorphism classes of simple modules. Let $\pi_i : A \to \mathrm{End}_F\ell_i$ for $1 \leq i \leq k$ be the algebra homomorphisms defining the actions of $A$ on $\ell_i$. Then their product*

$$\pi : A \to \mathrm{End}_F\ell_1 \times \cdots \times \mathrm{End}_F\ell_k \tag{1}$$

*is an isomorphism onto $\mathrm{End}_{\mathbb{D}_1}\ell_1 \times \cdot \times \mathrm{End}_{\mathbb{D}_i}\ell_k$, where $\mathbb{D}_i$ are the division rings $\mathrm{End}_A\ell_i$. In particular, if $F$ is algebraically closed, $\pi$ is a bijection.*

Proof: That $\pi$ is an injection is quite easy to see. Write $A = I_1 \oplus \cdots \oplus I_k$, where $I_j$ is the isotypical components of $A$ corresponding to $\ell_j$. If $a$ is in the kernel of $\pi$, then it kills all $I_j$ and so all of $A$, and hence is zero (since $A$ has identity). It remains only to show that the image of $\pi$ is the product of the images of $\pi_i$: note that $\pi_i$ is onto $\mathrm{End}_F\ell_i$ in case $F$ is algebraically closed (Lemma 1) and that its image is $\mathrm{End}_{\mathbb{D}_i}\ell_i$ in general (§**??**). For this, it is enough to prove the claim:

$$\pi_i(I_j) = 0 \text{ for } j \neq i$$

Indeed, then $\pi_i(I_i) = \pi_i(A)$, and given arbitrary $b_i$ in $\pi_i(A)$, choosing $a_i \in I_i$ such that $\pi_i(a_i) = b_i$, we get $\pi(a_1 + \cdots + a_k) = (\pi_1(a_1), \ldots, \pi_k(a_k)) = (b_1, \ldots, b_k)$.

To prove the claim, we use the observation made at the beginning of this subsection. Each $I_j$ (defined as above) is preserved by $\mathrm{End}_A A$, and so under right multiplication by elements of $A$. In other words, the $I_j$ are two-sided ideals (not just left ideals). Thus $I_jI_i = 0$ for $j \neq i$, so $I_j \subseteq \mathrm{Ann}\,\ell_i$ for $j \neq i$, which is precisely the claim. □

We list some consequences:

- (**A criterion for simplicity**)A semisimple finite dimensional algebra is simple if and only if it admits precisely one simple module.
- (**Density**) Assume $F$ to be algebraically closed. Let $V_1, \ldots, V_n$ be pairwise non-isomorphic simple modules for a semisimple finite dimensional algebra $A$ over $F$. Given arbitrary linear transformations $\varphi_i \in \mathrm{End}_F V_i$, there exists $a$ in $A$ such that $\varphi_i$ is left multiplication of $a$ on $V_i$.

- (**left semisimple is right semisimple**) The opposite of a finite dimensional semisimple algebra is semisimple. In particular, the notions of left and right semisimplicity coincide for finite dimensional algebras.

[ss:oreps]

3.3. **Applications to the ordinary representation theory of a finite group.** The "ordinary" in the title refers to the fact that the base field is algebraically closed of characteristic zero. Let $F$ be such a field, e.g., that of complex numbers. Let $G$ be a finite group. The group ring $FG$ of $G$ with coefficients in $F$ evidently has $F$-dimension $|G|$ (the cardinality of $G$). Moreover, it is semisimple by Maschke's theorem (§**??**). Wedderburn's theorem therefore applies: if $V_1, \ldots, V_k$ be simple modules such that no two are isomorphic and together represent all simple isomorphism classes, then $FG \simeq \mathrm{End}_F V_1 \times \cdots \times \mathrm{End}_F V_k$.

- Equating the dimensions of the centres on both sides, we see that there are as many simple isomorphism classes as conjugacy classes in the group.
- Equating $F$-dimensions on both sides, we get $|G| = (\dim V_1)^2 + \cdots + (\dim V_k)^2$.

## 4. The commutant of a semisimple algebra

[s:comm]

We begin with a simple but crucial observation. Let $F$ be a field and $V$, $W$ finite dimensional vector spaces over $F$. Consider the subalgebras $\mathrm{End}_F V$ and $\mathrm{End}_F W$ of $\mathrm{End}_F(V \otimes W)$. As can be easily checked, they are *commutants* of each other:

$$\begin{array}{l}\mathrm{End}_F W \text{ is precisely the subset of those elements of } \mathrm{End}_F(V \otimes W) \\ \text{that commute with } \mathrm{End}_F V \text{ (and vice versa).}\end{array} \tag{2}$$

[ss:schur]

4.1. **Commutant and bicommutant, after Schur.** Let now $F$ be algebraically closed, $V$ a finite dimensional $F$-vector space, and $A$ a semisimple subalgebra of $\mathrm{End}_F V$. Then $V$ is semisimple as an $A$-module (every $A$-module is semisimple). Following Schur, we pose:

$$\text{Can we identify the commutant } C \ (:= \mathrm{End}_A V) \text{ of } A \text{ inside } \mathrm{End}_F V? \tag{3}$$

Towards an answer to the above, let $\ell_1, \ldots, \ell_k$ be minimal left ideals of $A$ so chosen that no two are isomorphic as $A$-modules and together they represent all simple isomorphism classes of $A$-modules. Rewrite the isotypical decomposition $\ell_1^{\oplus r_1} \oplus \cdots \oplus \ell_k^{\oplus r_k}$ of $V$ as

$$V = \ell_1 \otimes \rho_1 \oplus \cdots \oplus \ell_k \otimes \rho_k \tag{4}$$

where $\rho_1, \ldots, \rho_k$ are $F$-vector spaces of respective dimensions $r_1, \ldots, r_k$, and $A$ acts on $\ell_i \otimes \rho_i$ by $a(x \otimes y) = ax \otimes y$.

Let $S$ denote the subalgebra $\mathrm{End}_F(\ell_1 \otimes \rho_1) \times \cdots \times \mathrm{End}_F(\ell_k \otimes \rho_k)$ of $\mathrm{End}_F V$. The image of $A$ in $\mathrm{End}_F V$ (under the map that defines $V$ as an $A$-module) is contained in $S$: it is in fact $A_1 \times \cdots \times A_k$, where $A_i$ denotes the image of $\mathrm{End}_F \ell_i$ in $\mathrm{End}_F(\ell_i \otimes \rho_i)$. Since $\mathrm{End}_A V$ preserves the isotypical $A$-components of $V$, it is a subalgebra of $S$. From observation (2) it now follows that it is $C_1 \times \cdots \times C_k$ where $C_i$ denotes the image of $\mathrm{End}_F \rho_i$ in $\mathrm{End}_F(\ell_i \otimes \rho_i)$ (which, let us note, is isomorphic to $\mathrm{End}_F \rho_i$).

Thus the commutant $C$ is isomorphic to $\mathrm{End}_F \rho_1 \times \cdots \times \mathrm{End}_F \rho_k$. It is therefore semisimple (§3.1). Moreover there is a bijective correspondence $\ell_i \leftrightarrow \rho_i$ between simple isomorphism classes of $A$ and those of $C$. Since $A$ and $C$ commute, we may consider $V$ as an $A \otimes C$-module with $(a \otimes c)v = a(cv) = c(av)$. The decomposition (4) of $V$ is also its isotypic decomposition as an $A \otimes C$-module: each $\ell_i \otimes \rho_i$ is simple (Corollary 3) and occurs exactly once. This last fact is expressed by saying that $V$ is *multiplicity free* as an $A \otimes C$-module. Note that out of the $k^2$ simple modules of $A \otimes C$ (namely $\ell_i \otimes \rho_j$), only $k$ appear in $V$ (those of the form $\ell_i \otimes \rho_i$, which defines the bijective correspondence).

Finally, observe that $A$ is the commutant of $C$: this follows from the same kind of reasoning that we used to identify $C$ explicitly. In other words, $A$ is its own *bicommutant*.

[ss:commex] 4.2. **Examples.** `At least two examples to be included here: schur-weyl, cauchy`

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