TWO WORKED OUT EXAMPLES OF ROTATIONS USING QUATERNIONS

This note is an attachment to the article "Rotations and Quaternions" which in turn is a companion to the video of the talk by the same title.

Example 1. Determine the image of the point $(1, -1, 2)$ under the rotation by an angle of $60°$ about an axis in the yz-plane that is inclined at an angle of $60°$ to the positive y-axis.

SOLUTION: The unit vector $\mathbf{u}$ in the direction of the axis of rotation is $\cos 60°\mathbf{j} + \sin 60°\mathbf{k} = \frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$. The quaternion (or vector) corresponding to the point $p = (1, -1, 2)$ is of course $p = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. To find the image of p under the rotation, we calculate qpq^{-1} where q is the quaternion $\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{u}$ and θ the angle of rotation ($60°$ in this case). The resulting quaternion—if we did the calculation right—would have no constant term and therefore we can interpret it as a vector. That vector gives us the answer.

We have $q = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{u} = \frac{\sqrt{3}}{2} + \frac{1}{4}\mathbf{j} + \frac{\sqrt{3}}{4}\mathbf{k} = \frac{1}{4}(2\sqrt{3} + \mathbf{j} + \sqrt{3}\mathbf{k})$. Since q is by construction a unit quaternion, its inverse is its conjugate: $q^{-1} = \frac{1}{4}(2\sqrt{3} - \mathbf{j} - \sqrt{3}\mathbf{k})$. Now, computing qp in the routine way, we get

$$qp = \frac{1}{4}((1 - 2\sqrt{3}) + (2 + 3\sqrt{3})\mathbf{i} - \sqrt{3}\mathbf{j} + (4\sqrt{3} - 1)\mathbf{k})$$

and then another long but routine computation gives

$$qpq^{-1} = \frac{1}{8}((10 + 4\sqrt{3})\mathbf{i} + (1 + 2\sqrt{3})\mathbf{j} + (14 - 3\sqrt{3})\mathbf{k})$$

The point corresponding to the vector on the right hand side in the above equation is the image of $(1, -1, 2)$ under the given rotation. Explicitly, that point is

$$\left(\frac{10 + 4\sqrt{3}}{8}, \frac{1 + 2\sqrt{3}}{8}, \frac{14 - 3\sqrt{3}}{8}\right) \qquad \square$$

Example 2. Consider the rotation by $60°$ about an axis in the xz-plane inclined at an angle of $60°$ to the positive x-axis. Determine the composition of this rotation with that in the earlier example: the one in the earlier example acts first.

SOLUTION: We need only compute $q'q$ where q is as in the previous example and $q' = \cos 30° + \sin 30°\mathbf{v}$ where $\mathbf{v}$ is the unit vector in the direction of the axis of the rotation specified in this example. We have $\mathbf{v} = \cos 60°\mathbf{i} + \sin 60°\mathbf{k} = \frac{1}{2}(\mathbf{i} + \sqrt{3}\mathbf{k})$. So $q' = \frac{\sqrt{3}}{2} + \frac{1}{4}(\mathbf{i} + \sqrt{3}\mathbf{k}) = \frac{1}{4}(2\sqrt{3} + \mathbf{i} + \sqrt{3}\mathbf{k})$. Now a long but routine calculation gives:

$$\begin{aligned} q'q &= \frac{1}{4}(2\sqrt{3} + \mathbf{i} + \sqrt{3}\mathbf{k}) \cdot \frac{1}{4}(2\sqrt{3} + \mathbf{j} + \sqrt{3}\mathbf{k}) \\ &= \frac{1}{16}(9 + \sqrt{3}\mathbf{i} + \sqrt{3}\mathbf{j} + 13\mathbf{k}) \end{aligned}$$

We now need to determine an angle θ and a unit vector $\mathbf{v}$ so that

$$\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{v} = \frac{1}{16}(9 + \sqrt{3}\mathbf{i} + \sqrt{3}\mathbf{j} + 13\mathbf{k})$$

Let us assume $0 \leq \frac{\theta}{2} \leq 180°$, so that the value of $\cos\frac{\theta}{2}$ (which is $\frac{9}{16}$ in our case) determines $\frac{\theta}{2}$. Then $\sin\frac{\theta}{2}$ is non-negative and in the present case we get $\sin\frac{\theta}{2} = \frac{5\sqrt{7}}{16}$. Now we determine $\mathbf{v}$:

$$\begin{aligned} \mathbf{v} = \frac{1}{\sin\frac{\theta}{2}} \cdot \frac{1}{16}(\sqrt{3}\mathbf{i} + \sqrt{3}\mathbf{j} + 13\mathbf{k}) &= \frac{1}{5\sqrt{7}}(\sqrt{3}\mathbf{i} + \sqrt{3}\mathbf{j} + 13\mathbf{k}) \\ &= \frac{1}{35}(\sqrt{21}\mathbf{i} + \sqrt{21}\mathbf{j} + 13\sqrt{7}\mathbf{k}) \end{aligned}$$

The composite rotation is by the angle θ around the axis determined by the unit vector $\mathbf{v}$. $\square$

The association $p \mapsto qpq^{-1}$, where p is a point in 3-space interpreted as a vector and q a unit quaternion, is what enables us to attach rotations to unit quaternions. Since the association does not change if we replace q by $-q$, it follows that both q and $-q$ are attached to the same rotation. Moreover, q and $-q$ are the only ones that have the same association. Indeed, if $qpq^{-1} = q'pq'^{-1}$ for all p, then qq'^{-1} commutes with all quaternions, and so has no imaginary component; being a unit quaternion as well, it can only be ± 1; thus $q' = \pm q$.