Finite dimensional C^*-algebras

S. Sundar

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Throughout $\mathcal{H}$, $\mathcal{K}$ stand for finite dimensional Hilbert spaces.

1 Spectral theorem for self-adjoint opertors

Let $A \in B(\mathcal{H})$ and let $\{\xi_1, \xi_2, \cdots, \xi_n\}$ be an orthonormal basis. Write

$$A\xi_j = \sum_{i=1}^{n} a_{ij}\xi_i.$$

Then (a_{ij}) is called the matrix of A w.r.t. the orthonormal basis $\{\xi_1, \xi_2, \cdots, \xi_n\}$. Note that $a_{ij} =< A\xi_j, \xi_i >$.

Exercise 1.1 *Let $A \in B(\mathcal{H})$. Prove that there exists a unique element $A^* \in B(\mathcal{H})$ such that for $\xi, \eta \in \mathcal{H}$,*

$$< A\xi, \eta >=< \xi, A^*\eta > .$$

The unique element A^* is called the adjoint of A.

Hint: What is the matrix of A^* ?

Exercise 1.2 *Let $A \in B(\mathcal{H})$. Let (a_{ij}) and (α_{ij}) be the matrices associated to A and A^* respectively w.r.t an orthonormal basis $\{\xi_1, \xi_2, \cdots, \xi_n\}$. Prove that $\alpha_{ij} = \overline{a_{ji}}$.*

Exercise 1.3 *Prove that $(AB)^* = B^*A^*$, $A^{**} = A$, $(\alpha A + \beta B)^* = \bar{\alpha}A^* + \bar{\beta}B^*$.*

Exercise 1.4 *Let $W \subset \mathcal{H}$ be a subspace and let $W^\perp := \{\eta \in \mathcal{H} :< \xi, \eta >= 0 \;\; for \;\; \xi \in W\}$. Prove that*

1. *$W + W^\perp = \mathcal{H}$ and $W \cap W^\perp = \{0\}$.*
2. *Given $\chi \in \mathcal{H}$, there exists unique $\xi \in W$ and $\eta \in W^\perp$ such that $\chi = \xi + \eta$.*

Definition 1.1 *Let $A \in B(\mathcal{H})$. The operator A is said to be **self-adjoint** if $A = A^*$.*

Example 1.2 *Let $W \subset \mathcal{H}$ be a subspace and $W^\perp$ be its orthogonal complement. For $\chi \in \mathcal{H}$, write $\chi = \xi + \eta$ with $\xi \in W$ and $\eta \in W^\perp$. Define a map $P_W : \mathcal{H} \to \mathcal{H}$ by $P_W(\chi) = \xi$. Prove that*

1. *The map P_W is linear, $P_W = id$ on W and $P_W = 0$ on $W^\perp$,*
2. *P_W is self-adjoint i.e. $P_W^* = P_W$,*
3. *P_W is an idempotent i.e. $P_W^2 = P_W$, and*
4. *The image of P_W is W.*

Definition 1.3 *An operator $P \in B(\mathcal{H})$ is called a **projection** if $P^2 = P = P^*$.*

Exercise 1.5 *Let $P \in B(\mathcal{H})$ be a projection. Let $W = Im(P)$. Prove that*

1. *$W = Ker(1 - P)$ and $W^\perp = Ker(P)$,*
2. *$P|_W = id$, $P|_{W^\perp} = 0$, and $P = P_W$.*

Proposition 1.4 *The map*

$$\{ \text{ subspaces of } \mathcal{H}\} \ni W \to P_W \in \{ \text{ set of projections in } B(\mathcal{H})\}$$

is a bijection.

Thus we think of projections and subspaces as one and the same.

Exercise 1.6 *Let W and V be subspaces of $\mathcal{H}$ and let P and Q be the corresponding projections. Prove that the following are equivalent.*

1. *W and V are orthogonal i.e. $< W, V >= 0$.*
2. *$PQ = 0$.*

Exercise 1.7 *Let $W_1, W_2, \cdots, W_n$ be subspaces of $\mathcal{H}$ and denote the corresponding projections by $P_1, P_2, \cdots, P_n$. Prove that the following are equivalent.*

1. *$\oplus_{i=1}^n W_i = \mathcal{H}$ i.e. $W_1 + W_2 + \cdots + W_n = \mathcal{H}$ and $< W_i, W_j >= 0$.*
2. *$P_i P_j = 0$ and $\sum_{i=1}^n P_i = 1$.*

Exercise 1.8 *Let $W \subset \mathcal{H}$ be a subspace and denote the orthogonal projection onto W by P. Consider a self-adjoint operator $A \in B(\mathcal{H})$. Then the following are equivalent.*

- *The operator A leaves W invariant i.e. $AW \subset W$.*
- *$PAP = AP$.*
- *A commutes with P i.e. $AP = PA$.*

Theorem 1.5 (Spectral theorem) *Let $A \in B(\mathcal{H})$ be a self-adjoint operator. Then there exists an orthonormal basis $\{\xi_1, \xi_2, \cdots, \xi_n\}$ and scalars $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ such that $A\xi_j = \lambda_j \xi_j$. (That is, the matrix of A is diagonal w.r.t. the orthonormal basis $\{\xi_1, \cdots, \xi_n\}$.)*

Proof. Let λ be an eigen value of A i.e. there exists a non-zero vector ξ such that $A\xi = \lambda\xi$. Then A leaves the one-dimensional subspace $W :=$ span $\{\xi\}$ invariant. Now apply Exercise 1.8 to conclude that A leaves $W^{\perp}$ invariant. By induction, there exists an orthonormal basis $\{\xi_2, \xi_3, \cdots, \xi_n\}$ of $W^{\perp}$ and scalars $\lambda_2, \lambda_3, \cdots, \lambda_n$ such that $A\xi_j = \lambda_j \xi_j$ for $j = 2, 3, \cdots, n$. Now set $\xi_1 = \xi$ and $\lambda_1 = \lambda$. Since $W \oplus W^{\perp} = \mathcal{H}$, it follows that $\mathcal{B} := \{\xi_1, \xi_2, \cdots, \xi_n\}$ is an orthonormal basis of $\mathcal{H}$ and the matrix of A w.r.t. $\mathcal{B}$ is diagonal. This completes the proof. □

Definition 1.6 *Let $A \in B(\mathcal{H})$. Define*

$$\sigma(A) := \{\lambda \in \mathbb{C} : \ A - \lambda \textit{ is not invertible}\}.$$

The set $\sigma(A)$ is called the ***spectrum*** *of A.*

Exercise 1.9 *Let $A \in B(\mathcal{H})$. Prove that the following are equivalent.*

1. *$\lambda \in \sigma(A)$.*
2. *There exists $\xi \in \mathcal{H}$ such that $\xi \neq 0$ and $A\xi = \lambda\xi$.*

For $\lambda \in \sigma(A)$, let $\mathcal{H}_\lambda = \{\xi \in \mathcal{H} : A\xi = \lambda\xi\}$.

Exercise 1.10 *Let $A \in B(\mathcal{H})$ and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be distinct elements of $\sigma(A)$. If $\xi_j \in \mathcal{H}_{\lambda_j}$ are non-zero vectors then prove that $\{\xi_1, \xi_2, \cdots, \xi_n\}$ are linearly independent. Conclude that $\sigma(A)$ is finite.*

The following exercise shows that $\sigma(A)$ is non-empty.

Exercise 1.11 *Let $\xi \in \mathcal{H}$ be a non-zero vector. Consider the set of vectors $\{\xi, A\xi, A^2\xi, \cdots,\}$. Since $\mathcal{H}$ is finite dimensional, conclude that there exists a polynomial p with complex co-efficients such that $p(A)\xi = 0$. Now factorise p as*

$$p(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n).$$

Use induction to show that there exists i such that $\lambda_i \in \sigma(A)$.

Proposition 1.7 *For $A \in B(\mathcal{H})$, $\sigma(A)$ is non-empty and finite.*

Exercise 1.12 *If the matrix of A w.r.t an orthonormal basis is $diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ what is $\sigma(A)$?*

Exercise 1.13 *If A is self-adjoint, prove that $\sigma(A) \subset \mathbb{R}$.*

Hint: Diagonalise A and use the fact that the matrices for A and A^* are equal.

Exercise 1.14 *Let A be self-adjoint and $\lambda, \mu \in \sigma(A)$. If $\lambda \neq \mu$ then $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ are orthogonal.*

Hint: Let $\xi \in \mathcal{H}_\lambda$ and $\eta \in \mathcal{H}_\mu$ be given. Consider the inner product $< A\xi, \eta > = < \xi, A\eta >$. □

Theorem 1.8 (Spectral theorem - A reformulation) *Let $A \in B(\mathcal{H})$ be self-adjoint. Then $\mathcal{H} = \oplus_{\lambda \in \sigma(A)} \mathcal{H}_\lambda$. Denote the orthogonal projection onto $\mathcal{H}_\lambda$ by P_λ. We have*

1. *If $\lambda \neq \mu$ then $P_\lambda P_\mu = 0$,*
2. $\sum_{\lambda \in \sigma(A)} P_\lambda = 1$, *and*
3. $A = \sum_{\lambda \in \sigma(A)} \lambda P_\lambda$.

Also note that for $\lambda \in \sigma(A)$,

$$P_\lambda = \prod_{\mu \in \sigma(A) - \{\lambda\}} \frac{(A - \mu)}{(\lambda - \mu)}.$$

Proposition 1.9 (Simultaneous diagonalisation) *Let $A, B \in B(\mathcal{H})$ be self-adjoint such that $AB = BA$. Then there exists an orthonormal basis $\mathcal{B} := \{\xi_1, \xi_2, \cdots, \xi_n\}$ such that the matrices of A and B w.r.t. $\mathcal{B}$ are diagonal.*

Proof. Write $\mathcal{H} = \oplus_{\lambda \in \sigma(A)} \mathcal{H}_\lambda$. Since A and B commute, B leaves $\mathcal{H}_\lambda = ker(A - \lambda)$ invariant. Now diagonalise B on each $\mathcal{H}_\lambda$ and complete the proof. □.

Exercise 1.15 *Extend Exercise 1.9 to any finite (or even infinite) number of self-adjoint operators.*

Definition 1.10 *Let $A \in B(\mathcal{H})$. The operator A is called **normal** if $A^*A = AA^*$.*

Theorem 1.11 (Spectral theorem for normal operators) *Let A be a normal operator on $\mathcal{H}$. Then there exists an orthonormal basis $\{\xi_1, \xi_2, \cdots, \xi_n\}$ and scalars $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ such that $A\xi_j = \lambda_j \xi_j$.*

Proof. Let $B := \frac{A+A^*}{2}$ and $C := \frac{A-A^*}{2i}$. Note that B and C are self-adjoint and $A = B + iC$. Also observe that $BC = CB$. Now apply Proposition 1.9 to complete the proof. □

2 Finite dimensional $*$-algebras

Definition 2.1 *An algebra $\mathcal{A}$ over $\mathbb{C}$*

1. *is a vector space over $\mathbb{C}$,*
2. *has an associative multiplicative structure $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. We write $m(a, b)$ as ab for $a, b \in \mathcal{A}$. Associativity means $(ab)c = a(bc)$.*

Also the mutliplication is compatible with the vector space structure. More precisely, we have

$$a(b + c) = ab + ac$$
$$(a + b)c = ac + bc$$
$$(\lambda a)b = \lambda(ab) = a(\lambda b)$$

We say $\mathcal{A}$ is unital if there exists an element $e \in \mathcal{A}$ such that $ae = ea = a$. (If such an element exists then it is unique.) We call e the multiplicative identity of $\mathcal{A}$ and denote it by 1.

Remark 2.2 *An algebra $\mathcal{A}$ is called **commutative** if $ab = ba$ for $a, b \in \mathcal{A}$.*

Definition 2.3 *Let $\mathcal{A}$ be an algebra. By a $*$-structure on $\mathcal{A}$, we mean a map $* : \mathcal{A} \to \mathcal{A}$, (we denote $*a$ by a^*), satisfying the following.*

1. *$*$ is an involution i.e. $(a^*)^* = a$.*
2. *$*$ is antimultiplicative i.e. $(ab)^* = b^*a^*$.*
3. *$*$ is conjugate linear i.e. $(a + b)^* = a^* + b^*$ and $(\lambda a)^* = \bar{\lambda}a^*$.*

An algebra $\mathcal{A}$ together with a $*$-structure is called a ***$*$-algebra**.*

Example 2.4 *Let $\mathcal{H}$ be a finite dimensional Hilbert space. Then $B(\mathcal{H})$ is a $*$-algebra.*

Example 2.5 *The matrix algebra $M_n(\mathbb{C})$ is a $*$-algebra. The multiplication is just the matrix multiplication. The $*$-structure is defined as follows:*

If $A = (a_{ij})$ then $A^* = (\alpha_{ij})$ where $\alpha_{ij} = \overline{a_{ji}}$.

Exercise 2.1 *Consider the set of $n \times n$ matrices. For $i, j = 1, 2, \cdots, n$, let e_{ij} be the matrix whose $(i, j)^{th}$ entry is 1 and the rest of the entries are 0. Prove the following.*

$$e_{ij}e_{kl} = \delta_{jk}e_{il}$$
$$e_{ij}^* = e_{ji}$$

Also show that $\{e_{ij}\}$ is a basis for $M_n(\mathbb{C})$. (Here $\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ if $k \neq l$.)

Example 2.6 *Let X be a non-empty set. Let $C(X) := \{f : X \to \mathbb{C}\}$. Then $C(X)$ is a $*$-algebra. The multiplication is pointwise multiplication and the $*$-structure is the complex conjugation i.e.*

$$(f.g)(x) = f(x)g(x)$$
$$f^*(x) = \overline{f(x)}$$

Prove that $C(X)$ is finite dimensional if and only if X is finite. Exhibit a basis for $C(X)$. Also note that $C(X)$ is commutative.

Example 2.7 (Group algebras) *Let G be a finite group. Let*

$$C[G] := \{f : G \to \mathbb{C}\}.$$

For $f, g \in C[G]$, let $f * g \in C[G]$ be defined by

$$(f * g)(s) := \sum_{t \in G} f(t)g(t^{-1}s).$$

The involution is defined as: For $f \in C[G]$, let

$$f^*(s) := \overline{f(s^{-1})}.$$

Then $C[G]$ is $*$-algebra and is called the group algebra of G.

For $g \in G$, let $\delta_g \in C[G]$ be given by

$$\delta_g(h) \;:=\; \begin{cases} 0 & if \quad g \neq h \\ 1 & if \quad g = h \end{cases}$$

Prove that $\delta_g^* = \delta_{g^{-1}}$ and $\delta_g * \delta_h = \delta_{gh}$. Also note that $\{\delta_g : g \in G\}$ forms a basis for $C[G]$. Thus $C[G] = \{\sum_{g \in G} a_g \delta_g : a_g \in \mathbb{C}\}$.

The multiplication on $C[G]$ is inherited from the group multiplication and the involution is from the group inversion.

Exercise 2.2 *Is $C[G]$ unital ? If so, what is the multiplicative identity ?*

Exercise 2.3 *Prove that $C[G]$ is commutative if and only if G is abelian.*

3 Representation of $*$-algebras

Definition 3.1 *Let $\mathcal{A}$ be a $*$-algebra. A representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $\pi : \mathcal{A} \to B(\mathcal{H})$ such that*

- *The map π is linear,*
- *π preserves multiplication i.e. $\pi(ab) = \pi(a)\pi(b)$, and*
- *π preserves involution i.e. $\pi(a^*) = (\pi(a))^*$.*

*If $\mathcal{A}$ is unital then π is called **unital** if $\pi(1) = 1$. The representation π is called **faithful** if π is injective.*

Definition 3.2 *Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be representations of $\mathcal{A}$. We say they are **unitarily equivalent** if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(a)U^* = \pi_2(a)$ for every $a \in \mathcal{A}$.*

Regular representation of a group: Let G be a finite group and consider the group algebra $C[G]$. Consider the vector space

$$\ell^2(G) := \{f : G \to \mathbb{C}\}.$$

For $\phi, \psi \in \ell^2(G)$, let $< \phi, \psi > := \sum_{g \in G} \phi(g)\overline{\psi(g)}$. Show that $<,>$ is an inner product. For $g \in G$, let $\epsilon_g \in \ell^2(G)$ be defined by

$$\epsilon_g(h) \;:=\; \begin{cases} 0 & if \quad g \neq h \\ 1 & if \quad g = h \end{cases}$$

Show that $\{\epsilon_g : g \in G\}$ is an orthonormal basis for $\ell^2(G)$.

For $g \in G$, let U_g be the unitary on $\ell^2(G)$ be defined by $U_g(\epsilon_h) = \epsilon_{gh}$. Observe that

$$U_g U_h = U_{gh}$$
$$U_g^* = U_{g^{-1}}$$

Now let $\pi : C[G] \to B(\ell^2(G))$ be defined by

$$\pi(\sum_{g \in G} a_g \delta_g) := \sum_{g \in G} a_g U_g.$$

Show that π is a faithful, unital $*$-representation and is called the regular representation of the group G.

GNS construction: For $\mathcal{A}$ be a unital $*$-algebra. For $a \in \mathcal{A}$, let $L_a : \mathcal{A} \to \mathcal{A}$ be defined by $L_a(b) = ab$. Then for every $a \in \mathcal{A}$, L_a is linear. Denote the set of linear maps on $\mathcal{A}$ by $\mathcal{L}(\mathcal{A})$.

Note that the map $\mathcal{A} \ni a \to L_a \in \mathcal{L}(\mathcal{A})$ is linear, multiplicative and unital. Assume that there exists an inner product structure on $\mathcal{A}$ such that the above map is a $*$-representation. What does it mean ? Let $<,>$ be an inner product on $\mathcal{A}$ such that $a \to L_a$ is a $*$-representation. Let $\tau : \mathcal{A} \to \mathbb{C}$ be defined by $\tau(a) :=< L_a(1), 1 >$.

Note that

$$\begin{aligned} < a, b > &=< L_a(1), L_b(1) > \\ &=< L_b^* L_a(1), 1 > \\ &=< L_{b^*a}(1), 1 > \\ &= \tau(b^*a) \end{aligned}$$

Thus the inner product is completely determined by the linear functional τ. We can turn this process around.

Definition 3.3 *Let $\mathcal{A}$ be a unital $*$-algebra and let $\tau : \mathcal{A} \to \mathbb{C}$ be a linear functional.*

1. *The functional τ is said to be* ***positive*** *if $\tau(a^*a) \geq 0$ for $a \in \mathcal{A}$.*

2. *The functional τ is called* ***unital*** *if $\tau(1) = 1$.*

3. *The functional τ is called* ***faithful*** *if $\tau(a^*a) = 0$ then $a = 0$.*

A positive, unital, linear functional on $\mathcal{A}$ is called a ***state****.*

Let $\mathcal{A}$ be a unital $*$-algebra and let τ be a faithful state on $\mathcal{A}$. Define an inner product on $\mathcal{A}$ by

$$< a, b >:= \tau(b^* a).$$

Verify that $<, >$ is an inner product and $\mathcal{A} \ni a \to L_a \in \mathcal{L}(\mathcal{A})$ is a unital $*$-representation. The representation thus obtained is called the GNS representation associated to the state τ.

Example 3.4 *Let G be a finite group. Consider the linear functional $\tau : C[G] \to \mathbb{C}$ defined by*

$$\tau(\sum_{g \in G} a_g g) := a_e.$$

Verify that τ is a state and is faithful. Prove that the GNS representation associated to τ is unitarily equivalent to the regular representation.

Definition 3.5 (Direct sum of representations) *Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be two representations of $\mathcal{A}$. The direct sum is the representation $(\pi_1 \oplus \pi_2, \mathcal{H}_1 \oplus \mathcal{H}_2)$.*

Let $\mathcal{A}$ be a $*$ algebra and $(\pi, \mathcal{H})$ be a representation. A subspace $W \subset \mathcal{H}$ is called an invariant subspace if for every $a \in \mathcal{A}$, $\pi(a)W \subset W$. If W is invariant then $(\pi|_W, W)$ is again a representation.

Exercise 3.1 *Let $(\pi, \mathcal{H})$ be a representation of $\mathcal{A}$ and $W \subset \mathcal{H}$ be a subspace. Denote the orthogonal projection onto W by P. Prove that the following are equivalent.*

1. *The subspace W is an invariant subspace.*
2. *For $a \in \mathcal{A}$, $\pi(a)P = P\pi(a)$.*

Conclude that if W is invariant then $W^\perp$ is invariant.

Exercise 3.2 *Let $(\pi, \mathcal{H})$ be a representation of $\mathcal{A}$ and $W \subset \mathcal{H}$ be an invariant subspace. Prove that $(\pi, \mathcal{H})$ is unitarily equivalent to $(\pi, W) \oplus (\pi, W^\perp)$.*

Definition 3.6 *Let $(\pi, \mathcal{H})$ be a representation of $\mathcal{A}$. We say that π is* ***irreducible*** *if the only invariant subspaces are 0 and $\mathcal{H}$.*

Example 3.7 *Prove that the canonical representation of $M_n(\mathbb{C})$ on $\mathbb{C}^n$ is irreducible.*

Proposition 3.8 *Let $(\pi, \mathcal{H})$ be a representation of $\mathcal{A}$. Prove that π is unitarily equivalent to a direct sum of irreducible representations.*

Hint: Use Ex. 3.2. □

4 Double commutant theorem

Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be $*$-representations of $\mathcal{A}$. Define

$$\mathcal{L}_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_2) := \{T \in B(\mathcal{H}_1, \mathcal{H}_2) : T\pi_1(a) = \pi_2(a)T \ \text{ for } a \in \mathcal{A}\}.$$

Observe that if $T \in \mathcal{L}_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_2)$ then $T^* \in \mathcal{L}_{\mathcal{A}}(\mathcal{H}_2, \mathcal{H}_1)$. Also observe that if $T \in \mathcal{L}_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_2)$ and $S \in \mathcal{L}_{\mathcal{A}}(\mathcal{H}_2, \mathcal{H}_3)$ then $ST \in \mathcal{L}_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_3)$.

Exercise 4.1 *If $(\pi, \mathcal{H})$ is a representation of $\mathcal{A}$, show that*

$$\mathcal{L}_{\mathcal{A}}(\mathcal{H}, \mathcal{H}) := \{T \in B(\mathcal{H}) : \ T\pi(a) = \pi(a)T \ \ \textit{for } a \in \mathcal{A}\}$$

is a unital $*$-algebra.

Definition 4.1 *Let $\mathcal{S} \subset B(\mathcal{H})$. Define its **commutant**, denoted $\mathcal{S}^{'}$, by*

$$\mathcal{S}^{'} := \{T \in B(\mathcal{H}) : Ts = sT \ \ \textit{for } s \in \mathcal{S}\}.$$

Prove that $\mathcal{S}^{'}$ is a linear subspace of $B(\mathcal{H})$, contains the identity operator $1_{\mathcal{H}}$ and is closed under multiplication. Also show that if $\mathcal{S}$ is $*$-closed then $\mathcal{S}^{'}$ is $*$-closed. Thus $\mathcal{S}^{'}$ is a $*$-algebra if $\mathcal{S}$ is $*$-closed.

Note that for a representation $(\pi, \mathcal{H})$ of $\mathcal{A}$, $\mathcal{L}_{\mathcal{A}}(\mathcal{H}, \mathcal{H})$ is the commutant $\pi(\mathcal{A})^{'}$.

Exercise 4.2 *Let $(\pi_i, \mathcal{H}_i)_{i=1}^n$ be representations of $\mathcal{A}$. Consider the direct sum representation $(\pi := \oplus_{i=1}^n \pi_i, \mathcal{H} = \oplus_{i=1}^n \mathcal{H}_i)$ of $\mathcal{A}$. Show that $\mathcal{L}_{\mathcal{A}}(\mathcal{H}) = \{(x_{ij}) : x_{ij} \in \mathcal{L}_{\mathcal{A}}(\mathcal{H}_j, \mathcal{H}_i)\}$.*

Lemma 4.2 *Let $\mathcal{A}$ be a unital $*$-algebra and $\pi : \mathcal{A} \to B(\mathcal{H})$ be a unital $*$-representation. Let $\pi_n := \underbrace{\pi \oplus \pi \oplus \cdots \pi}_{n \text{ times}}$. Then*

1. $\pi_n(\mathcal{A})^{'} = M_n(\pi(\mathcal{A})^{'})$, *and*
2. $\pi_n(\mathcal{A})^{''} = \Big\{ \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix} : \ x \in \pi(\mathcal{A})^{''}) \Big\}.$

Theorem 4.3 (von Neumann's double commutant theorem) *Let $\mathcal{A}$ be a unital $*$-algebra and let $\pi : \mathcal{A} \to B(\mathcal{H})$ be a unital $*$-representation. Then $\pi(\mathcal{A})^{''} = \pi(\mathcal{A})$.*

Proof. Clearly $\pi(\mathcal{A}) \subset \pi(\mathcal{A})''$. Let $x \in \pi(\mathcal{A})''$ be given.

Step 1: Let $\xi \in \mathcal{H}$ and $W := \{\pi(a)\xi : a \in \mathcal{A}\}$. Denote the orthogonal projection onto W by P. Since π is unital, $\xi \in W$ i.e. $P\xi = \xi$. Note that W is an invariant subspace for $\mathcal{A}$ and hence $P \in \pi(\mathcal{A})'$. Now $Px\xi = xP\xi = x\xi$. Thus $x\xi \in W$.

Thus we have shown that for every $\xi \in \mathcal{H}$, there exists $a \in \mathcal{A}$ such that $x\xi = \pi(a)\xi$.

Step 2: Let $\{\xi_1, \xi_2, \cdots, \xi_n\}$ be an orthonormal basis. Let

$$\pi_n = \underbrace{\pi \oplus \pi \oplus \cdots \oplus \pi}_{\text{n times}},$$

$$\mathcal{H}_n = \underbrace{\mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{\text{n times}},$$

$$\xi = \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n.$$

Apply Step 1 and Lemma 4.2 to show that there exists $a \in \pi(\mathcal{A})$ such that $x\xi_i = \pi(a)\xi_i$ for $i = 1, 2, \cdots, n$. Thus $x \in \pi(\mathcal{A})$. This completes the proof. □

Exercise 4.3 *Let $\mathcal{A}$ be a $*$-subalgebra of $B(\mathcal{H})$. Show that $\mathcal{A}$ is the linear span of projections in $\mathcal{A}$.*

Hint: First observe that $\mathcal{A}$ is the linear span of self-adjoint elements in $\mathcal{A}$. Now use spectral theorem (1.8).

Proposition 4.4 (Schur's lemma) *Let $(\pi, \mathcal{H})$ be an irreducible representation of $\mathcal{A}$. Then $\pi(A)' = \mathbb{C}1_{\mathcal{H}}$.*

Proof. It is enough to show that if $p \in \pi(A)'$ is a non-zero projection then $p = 1$. Now use Ex. 3.1 and use the fact that π is irreducible to conclude that $p = 1$. □

Proposition 4.5 (Schur's lemma) *Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be irreducible representations of $\mathcal{A}$. If $\mathcal{L}_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_2) \neq 0$ then π_1 and π_2 are unitarily equivalent.*

Proof. Suppose that $\mathcal{L}_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_2)$ be non-zero and let $T \in \mathcal{L}_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_2)$ be non-zero. Then $T^*T \neq 0$ and $T^*T \in \pi_1(A)'$. By Prop. 4.4, there exists $\lambda > 0$ (Justify!) such that $T^*T = \lambda$. Let $U = \frac{T}{\sqrt{\lambda}}$. Observe that $U \in \mathcal{L}_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_2)$ and U is unitary. This completes the proof. □

5 Structure of finite dimensional C^*-algebras

Definition 5.1 *Let $\mathcal{A}$ be a unital finite dimensional $*$-algebra. The algebra $\mathcal{A}$ is called a C^*-algebra if there exists a faithful representation $\pi : \mathcal{A} \to B(\mathcal{H})$.*

Exercise 5.1 *Let $\mathcal{A}$ be a finite dimensional C^*-algebra. Prove that there exists a unital faithful representation of $\mathcal{A}$.*

Hint: Let $\pi : \mathcal{A} \to B(\mathcal{H})$ be a faithful representation and let $P = \pi(1)$. Note that P is a projection and $Im(P)$ is invariant under π. Now complete the proof. $\square$

Exercise 5.2 *Prove that all the $*$-algebras we have considered so far are in fact C^*-algebras.*

Let $(A_i)_{i=1}^n$ be $*$-algebras. The direct sum

$$\oplus_{i=1}^n A_i := \{(a_1, a_2, \cdots, a_n) : a_i \in A_i\}$$

is a $*$-algebra with the multiplication and $*$-operation given by

$$(a_1, a_2, \cdots, a_n)(b_1, b_2, \cdots, b_n) = (a_1 b_1, a_2 b_2, \cdots, a_n b_n),$$
$$(a_1, a_2, \cdots, a_n)^* = (a_1^*, a_2^*, \cdots, a_n^*).$$

Exercise 5.3 *Show that if $(A_i)_{i=1}^n$ are C^*-algebras then the direct sum $\oplus_{i=1}^n A_i$ is a C^*-algebra.*

Now we can state our main theorem.

Theorem 5.2 *Let $\mathcal{A}$ be a finite dimensional C^*-algebra. Then $\mathcal{A}$ is isomorphic to a direct sum of matrix algebras i.e. there exists positive integers $m_1, m_2, \cdots, m_r$ such that*

$$\mathcal{A} \cong M_{m_1}(\mathbb{C}) \oplus M_{m_2}(\mathbb{C}) \oplus \cdots \oplus M_{m_r}(\mathbb{C}).$$

Moreover if

$$M_{m_1}(\mathbb{C}) \oplus M_{m_2}(\mathbb{C}) \oplus \cdots \oplus M_{m_r}(\mathbb{C}) \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_s}(\mathbb{C})$$

then $r = s$ and there exists a permutation σ of $\{1, 2, \cdots, r\}$ such that $n_i = m_{\sigma(i)}$.

Proof. Let $\pi : \mathcal{A} \to B(\mathcal{H})$ be a unital faithful representation. Split π into irreducible representations. Convince yourself that there exists inequivalent (i.e. not unitarily equivalent) irreducible representations $\pi_1, \pi_2, \cdots, \pi_r$ and positive integers $d_1, d_2, \cdots, d_r$ such that

$$\pi \cong \pi_1^{d_1} \oplus \pi_2^{d_2} \oplus \cdots \oplus \pi_r^{d_r}.$$

Now note that the unital representation $\rho := \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_r$ is faithful. Let $\mathcal{H}_i$ be the Hilbert space on which π_i acts and set $m_i := dim(\mathcal{H}_i)$.

Now use Ex. 4.2 and Schur's lemma to conclude that

$$\rho(\mathcal{A})^{'} = \left\{ \begin{pmatrix} \lambda_1 1_{\mathcal{H}_1} & 0 & \cdots & 0 \\ 0 & \lambda_2 1_{\mathcal{H}_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_r 1_{\mathcal{H}_r} \end{pmatrix} : \lambda_i \in \mathbb{C} \right\}.$$

Now show that

$$\rho(\mathcal{A})^{''} = \left\{ \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_r \end{pmatrix} : \quad x_i \in B(\mathcal{H}_i) \right\}.$$

Use Theorem 4.3 to conclude that

$$\mathcal{A} \cong \rho(\mathcal{A}) = \rho(\mathcal{A})^{''} \cong M_{m_1}(\mathbb{C}) \oplus M_{m_2}(\mathbb{C}) \oplus \cdots \oplus M_{m_r}(\mathbb{C})$$

Do the following exercises to show uniqueness. □

Definition 5.3 *Let $\mathcal{A}$ be a $*$-algebra. The center of $\mathcal{A}$ denoted $Z(\mathcal{A})$ is defined as*

$$Z(\mathcal{A}) := \{z \in \mathcal{A} : za = az \quad \text{for } a \in \mathcal{A}\}.$$

The center $Z(\mathcal{A})$ is a commutative C^*-algebra.

Definition 5.4 *If $p \in \mathcal{A}$ is a projection, the cutdown of $\mathcal{A}$ by p is the $*$-algebra $p\mathcal{A}p$ defined as*

$$p\mathcal{A}p := \{pap : a \in \mathcal{A}\}$$

Exercise 5.4 *Show that $Z(M_n(\mathbb{C})) = \mathbb{C}$.*

Exercise 5.5 *Show that $Z(\mathcal{A}_1 \oplus \mathcal{A}_2) = Z(\mathcal{A}_1) \oplus Z(\mathcal{A}_2)$.*

Exercise 5.6 *Let $\mathcal{A} = M_{m_1}(\mathbb{C}) \oplus M_{m_2}(\mathbb{C}) \oplus \cdots \oplus M_{m_r}(\mathbb{C})$. For $i = 1, 2, \cdots, r$, let*

$$z_i = (0, 0, \cdots, \underbrace{1}_{i^{th} place}, \cdots, 0).$$

Note that z_i is a projection and $z_i \in Z(\mathcal{A})$. Prove that $\{z_i : i = 1, 2, \cdots, r\}$ forms a basis for $Z(\mathcal{A})$. Also note that if $i \neq j$ then $z_i z_j = 0$. Prove that $r = dim(Z(\mathcal{A}))$ and $m_i^2 = dim(z_i \mathcal{A} z_i)$.

Let $\{e_i : i = 1, 2, \cdots, r\}$ be a set of mutually orthogonal central projections. Assume that they form a basis for $Z(\mathcal{A})$. Prove that $\{e_i : i = 1, 2, \cdots, r\} = \{z_i : i = 1, 2, \cdots, r\}$.

The central projections $\{z_i\}_{i=1}^n$ are called the minimal central projections of $\mathcal{A}$.

Exercise 5.7 *Complete the proof of Theorem 5.2.*

References

[Sun97] V. S. Sunder, *Functional analysis*, 1997, Spectral theory.