Partitions of integers and their higher-dimensional generalisations

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Outline

Introduction

The partition function

- A partition of an integer $n > 0$ is to express it as a sum of positive integers. For instance, $2 + 1 + 1$ is a partition of the integer 4.
- As $1 + 2 + 1$ and $2 + 1 + 1$ are the same partition, one chooses to write it as a *weakly decreasing* sequence $(2, 1, 1)$ to get a unique representative for each partition.
- Let $p(n)$ denote the number of partitions of n. For $n = 4$, one has

$$4 \quad 3\,1 \quad 2\,2 \quad 2\,1\,1 \quad 1\,1\,1\,1 \Longrightarrow \boxed{p(4) = 5}\,.$$

  $p(n)$ is called the partition function.
- What is $p(200)$? Do we need to enumerate all the partitions to get the count?

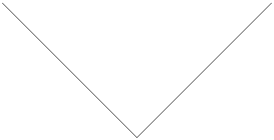
Refining the partition function

- Let $p(n|r \text{ parts})$ denote the number of partitions of n with r parts.
- Similarly, let $p(n|\text{l.p.} = r)$ denote the number of partitions of n with largest part r.
- For instance, $(2, 1, 1)$ is a partition of 4 with 3 parts and largest part 2.

partition	4	3 1	2 2	2 1 1	1 1 1 1
# of parts	1	2	2	3	4
l.p.	4	3	2	2	1

- Observe that $p(4|r \text{ parts}) = p(4|\text{l.p.} = r)$ for $r = 1, 2, 3, 4$.
- Euler: This is true for all n. How does one prove such a statement?

Let's play a game



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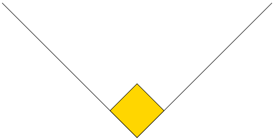

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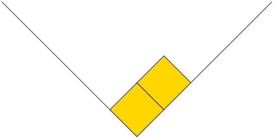

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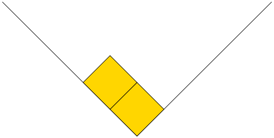

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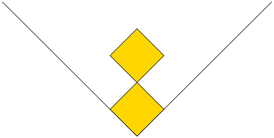

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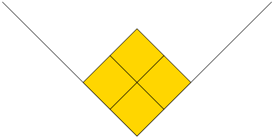

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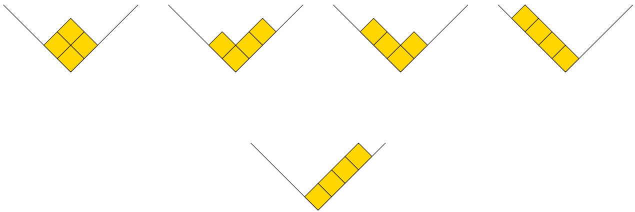

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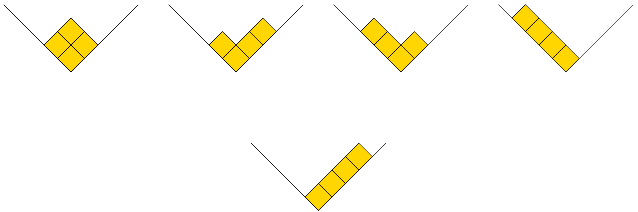
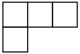
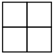


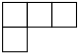
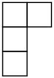
- Depending on your area of specialisation, you will recognise these to be Young diagrams (drawn Russian style) or directed compact lattice animals or clusters or tetris blocks.

From partitions to Young diagrams

- Given a partition, $(a_1, a_2, a_3, \ldots)$, draw a Young diagram with a_k-boxes in the k-th row. For instance,

$3\ 1 \quad \longleftrightarrow$  $\quad ; \quad 2\ 2 \quad \longleftrightarrow$ 

- It is clear that this map is a bijection. Given a Young diagram, one can obtain the corresponding partition by counting the number of boxes in each row.
- There is an involution that acts on Young diagrams called conjugation. It corresponds to the xy-flip.

$(3\ 1) =$  $\longleftrightarrow$  $(= 2\ 1\ 1)$

- Viewed as acting on partitions, we see that it maps a partition with r-parts to one with largest part r. This is Ferrer's bijective proof showing that $p(n|r \text{ parts}) = p(n|\text{l.p.} = r)$.

Higher Dimensional Partitions

Plane partitions

- MacMahon proposed the following generalisation of partitions. A two-dimensional or plane partition of a non-negative integer n is a two-dimensional array of non-negative integers $a_{i,j}$ $(i, j = 1, 2, \ldots)$ such that

$$\forall i,\, j \quad a_{i,j} \leq a_{i+1,\, j} \;,\; a_{i,j} \leq a_{i,j+1} \text{ and } \sum_{i,j} a_{i,j} = n \; .$$

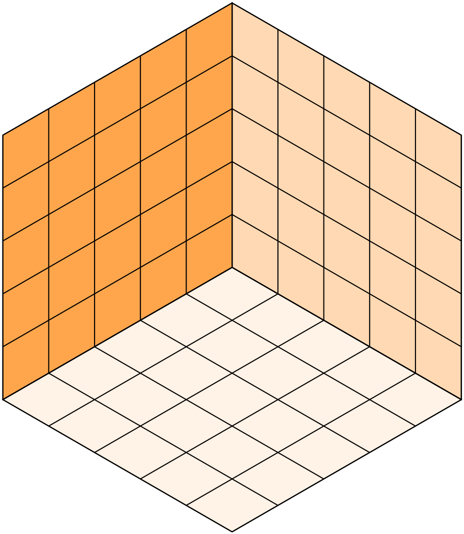
- Let $p_2(n)$ denote the number of plane partitions of n.
- For instance, the plane partitions of 4 are

$$4 \qquad 3\ 1 \qquad \begin{matrix} 3 \\ 1 \end{matrix} \qquad 2\ 2 \qquad \begin{matrix} 2 \\ 2 \end{matrix} \qquad 2\ 1\ 1 \qquad \begin{matrix} 2 & 1 \\ 1 & \end{matrix} \qquad \begin{matrix} 2 \\ 1 \\ 1 \end{matrix}$$

$$1\ 1\ 1\ 1 \qquad \begin{matrix} 1 & 1 & 1 \\ 1 & & \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 1 & \\ 1 & \end{matrix} \qquad \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \end{matrix} \qquad \Longrightarrow \boxed{p_2(4) = 13}$$

- It is easy to see that $p_2(n) \geq p(n)$ since every partition is plane partition.

Plane Partitions as 3d Young Diagrams



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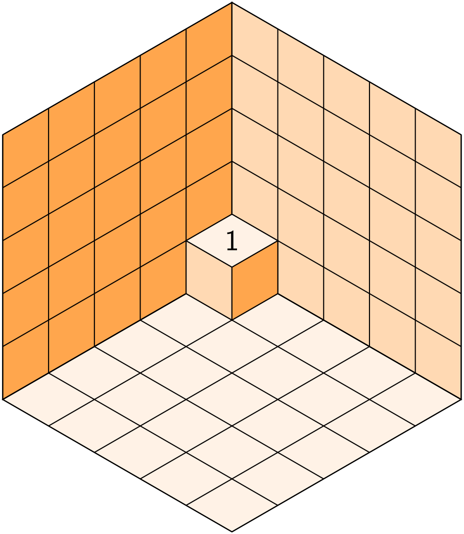

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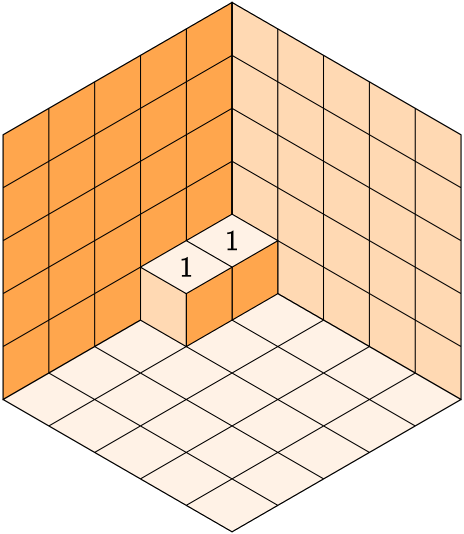

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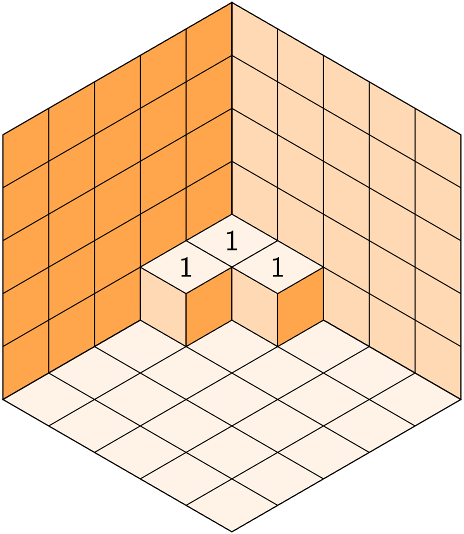

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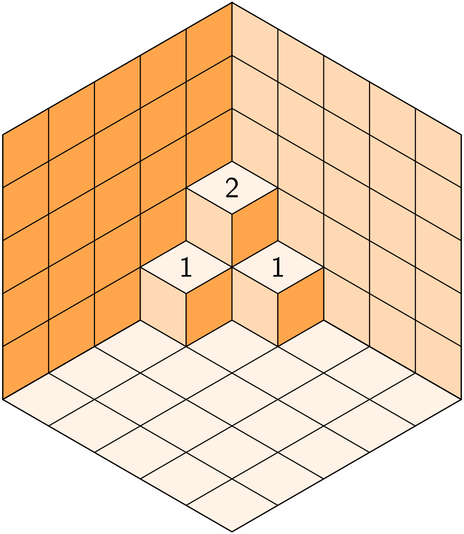

Ferrers diagrams

- It is clear how to generalise Young diagrams to higher dimensions replacing cubes in 3D to hypercubes in 4D and so on. However, visualistion is not possible.
- We replace the squares/cubes/... with integral points in $\mathbb{R}_+^{d+1}$ – call the points nodes.
- An unrestricted d-dimensional partition of n is a collection of n points (nodes) in $\mathbb{Z}_{\geq 0}^{d+1}$ satisfying the following property: if the collection contains a node $\mathbf{a} = (a_1, a_2, \ldots, a_{d+1})^T$, then all nodes $\mathbf{x} = (x_1, x_2, \ldots, x_{d+1})^T$ with $0 \leq x_i \leq a_i$ $\forall$ $i = 1, \ldots, d+1$ also belong to it.
- For instance, the following is a one-dimensional partition of 4

$$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\} \text{ or } \left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{smallmatrix}\right) \text{ in compressed form },$$

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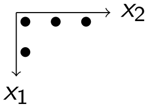
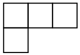
- The same collection of nodes can be viewed as a Ferrers diagram or a Young diagram.



or

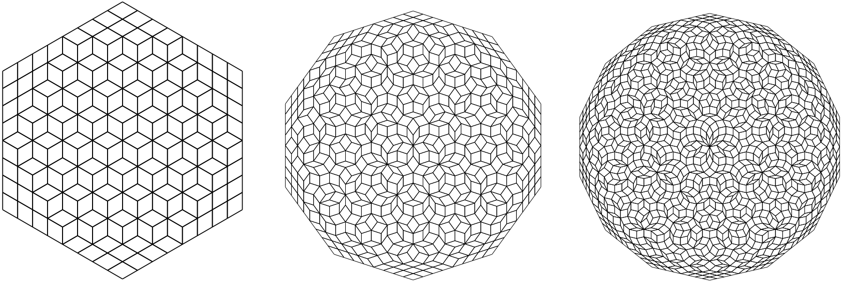


Where do these higher-dimensional partitions appear

Higher-dimensional partitions appear in several different areas of physics, mathematics and computer science. I list a few

- The infinite state Potts model in $(d+1)$ dimensions gets related to d-dimensional partitions in the high temp. limit;
- in the study of directed compact lattice animals;
- in the counting of BPS states in string theory and supersymmetric field theory.
- Bounded plane partitions appear as a domino tilings of hexagon or a dimer configurations or as perfect matchings of a bipartite graph.

Tilings



Projections of tilings from $D = 3, 5, 7$ to two-dimensions[1]. The $D = 3$ is the projection of the Young diagram for plane partitions to the plane leading to rhombus tilings of a hexagon. The other two examples corresponds to tiling with different kinds of tiles.

[1] Figure from Widom et. al. J. Stat. Phys. 120, 837 (2005).

Generating Functions and their applications

Generating Functions

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$$P(q) := \sum_{n=0}^{\infty} p(n)\ q^n\ .$$

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- We illustrate how Euler's formula works for $p(4)$

$$\begin{aligned} P(q) =& (1 + q + q^{1+1} + q^{1+1+1} + q^{1+1+1+1} + O(q^5)) \\ & \times (1 + q^2 + q^{2+2} + O(q^6)) \\ & \times (1 + q^3 + O(q^6)) \\ & \times (1 + q^4 + O(q^8)) \times (1 + O(q^5)) \end{aligned}$$

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Generating Functions – more examples

- The generating function of partitions whose parts $\leq M$ is

$$\frac{1}{\prod_{m=1}^{M}(1-q^m)}$$

- The generating function of partitions of n whose parts arise from a set S of positive integers is

$$1+\sum_{n=1}^{\infty}\sum_{r=1}^{n} p(n|\text{parts} \in S)\ q^n = \frac{1}{\prod_{m\in S}(1-q^m)}$$

- A two-variable generalisation

$$1+\sum_{n=1}^{\infty}\sum_{r=1}^{n} p(n|r\ \text{parts})\ q^n t^r = \frac{1}{\prod_{m=1}^{\infty}(1-t\ q^m)}$$

Towards a practical formula for $p(n)$ - I

- Let $p(n|\text{distinct parts})$ denote the number of partitions of n with distinct parts. Then, one has

$$1 + \sum_{n=1}^{\infty} p(n|\text{distinct parts})\ q^n = \prod_{m=1}^{\infty} (1 + q^m)\ .$$

-

$$1 + \sum_{n=1}^{\infty} \left(p(n|\text{even dist. parts}) - p(n|\text{odd dist. parts})\right)\ q^n$$
$$= \prod_{m=1}^{\infty} (1 - q^m)$$

This is the inverse of the generating function of partitions!

Towards a practical formula for $p(n)$ - II

Theorem (Euler's Pentagonal Theorem)

$$(p(n|\textit{even dist. parts}) - p(n|\textit{odd dist. parts})) = e(n)$$

$$\textit{where } e(n) = \begin{cases} (-1)^j & \textit{for } n = \frac{j(3j-1)}{2},\ j \in \mathbb{Z} \\ 0 & \textit{otherwise .} \end{cases}$$

$e(n) \neq 0$ when n is a generalised pentagonal numbers. We get

$$\left(1 + \sum_{m=1}^{\infty} e(m)\ q^m\right) \times \sum_{n=0}^{\infty} p(n)\ q^n = 1$$

leading to

$$p(n) = -\sum_{m=1}^{n-1} e(m)\ p(n-m)\ ,$$

which provides an efficient algorithm to compute $p(n)$ recursively.

The formula and its use

- The first few generalised pentagonal numbers are $1, 2, 5, 7, 12, 15, \ldots$ (sequence A001318 in OEIS)

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \ldots$$

- Major Percy MacMahon used the above formula and tabulated the first two hundred numbers and determined that

$$p(200) = 3972999029388 \sim 3.973 \times 10^{12} \ .$$

  Recall that this was before the advent of computers.
- Turns out one can compute $p(n)$ in time proportional to $n^{3/2}$. Can one do better?
- What is $p(10^{20})$? Needless to say this is out of computational reach using the above formula even today. However, the answer is known!

See George Andrews and Kimmo Eriksson, *Integer Partitions*, Cambridge University Press (2004).

Generating Function for Plane Partitions

- Define the generating function for a d-dimensional partition as

$$P_d(q) = 1 + \sum_{n=1} p_d(n)\ q^n\ .$$

- MacMahon conjectured and eventually[2] proved that the generating function for plane partitions is given by

$$P_2(q) = \prod_{m=1}^{\infty} (1 - q^m)^{-m}\ .$$

- One can use the above formula to obtain a nice recursive formula for the plane partition function

$$p_2(n) = \frac{1}{n} \sum_{k=1}^{n-1} \sigma_2(k)\ p_2(n-k)\ .$$

  However, there appears to be no analog of the pentagonal recursive formula that we saw for the partition function.

[2] 20 years later! No elementary proof is known till today.

Counting of BPS states (black hole microstates)

- The generating function of electrically charged $\frac{1}{2}$-BPS states in the het. string compactified on a six-torus is given by $(\frac{1}{2}\mathbf{q}_e^2 := n-1)$

$$\sum_{n=-1}^{\infty} d(n) q^n = \frac{16}{q\prod_{n=1}^{\infty}(1-q^n)^{24}} = \frac{16}{\eta(\tau)^{24}} \ .$$

- The generating function of Donaldson-Thomas (or Gopakumar-Vafa) invariants on the non-commut. conifold is given by ($q \sim e^{-g_s}$; t – Kähler modulus) [Szendroi,Young]

$$\prod_{n=1}^{\infty}(1-q^n)^{-n}(1-e^{-t}q^n)^{-n}(1-e^{+t}q^n)^{-n}$$

Note the appearance of the Euler and MacMahon generating functions.

Generating Functions for $d > 2$?

- Define the generating function for a d-dimensional partition as

$$P_d(q) = 1 + \sum_{n=1} p_d(n)\ q^n\ .$$

- MacMahon also conjectured formulae for the generating function for solid and other higher dimensional partitions. Let

$$M_d(q) := \prod_{m=1}^{\infty} (1 - q^m)^{-\binom{n+d-2}{d-1}} = 1 + \sum_{n=1}^{\infty} m_d(n)\ q^n\ .$$

  It was shown in 1967 by Atkin et. al. that this fails i.e., $m_d(n) \neq p_d(n)$ for $d > 2$ and $n \geq 6$.
- It appears that there is no simple formula for the generating function.

History of Higher Dimensional Partitions

After MacMahon, the first serious computation of higher dimensional partitions, due to Atkin, Bratley, MacDonald and McKay, appeared in 1967. Here is a report by Birch on this paper in his memoir on Atkin:

I cannot resist mention [1967d], on m-dimensional partitions. At the time the authors complained that no one seemed to know anything about them except in the first two cases (ordinary partitions corresponding to the case m=2, and the case m=3 more often known as plane partitions); and very little seems to have been discovered since; there is a note on the subject in [1971d]. In the words of the third author, the paper landed like a lead balloon; but they look genuinely interesting.

Stanley in his 1971 doctoral thesis writes:

The case $r = 2$ *has a well-developed theory – here 2-dimensional partitions are known as plane partitions. See 21 and the survey article by Stanley[34] for results on plane partitions. For* $r \geq 3$*, almost nothing is known and Proposition 11.1 casts only a faint glimmer of light on a vast darkness.*

Enumerating higher dimensional partitions

- One can attempt to carry out exact enumerations. There are two algorithms, one due to Bratley and McKay (1967) and another one due to Knuth (1970).
- Knuth's algorithm enumerates the number of topological sequences in a partially ordered set. Choosing the set is $\mathbb{N}^d$, the number of topological sequences of a given index can be used to obtain the number of d-dimensional partitions. He enumerated the first 28 numbers that were extended to 50 by Mustonen and Rajesh in 2002.
- In 2010, a undergraduate student at IITM, Srivatsan, created a non-trivial (parallel) extension of Knuth's algorithm. This has enabled us[3] to extend the exact enumeration to obtain the number of solid partitions of 72 (after half a million CPU hours)

$$p_3(72) = 3464274974065172792 \sim 3.464 \times 10^{18} \ .$$

Adding more numbers is not easy but you can help.

[3] http://boltzmann.wikidot.com/the-partitions-project

The asymptotics of higher dimensional partitions

- It was shown by Bhatia et. al. (1997) that
$$\lim_{n\to\infty} n^{-d/(d+1)} \log p_d(n) \to C_d \text{ , a constant.}$$
- It is know that $C_1 = 2(\zeta(2))^{1/2}$ and $C_2 = \frac{3}{2}(2\zeta(3))^{1/3}$ – these are derived from the generating function.
- Mustonen and Rajesh estimated the constant for solid partitions C_3 using Monte Carlo simulations and obtained $C_3 = 1.78 \pm 0.01$, a value that is compatible with the asymptotic values for $m_3(n)$. One has
$$\lim_{n\to\infty} n^{-3/4} \log m_3(n) \to \frac{4}{3}(3\zeta(4))^{1/4} \sim 1.7898 \ .$$
They conjectured that the MacMahon numbers capture the leading asymptotic behaviour of solid partitions.
- Recent work[4] show that $C_3 = 1.822 \pm 0.001$ which, disappointly, disproves the conjecture. There appears to be no glimmer of light!

[4] Destainville and SG, 2014 http://arxiv.org/abs/1406.5605.

The Hardy-Ramanujan-Rademacher formula

The HRR formula

- On can invert the generating function $P(q)$

$$p(n) = \frac{1}{2\pi i} \int_{\mathcal{C}_\rho} \mathrm{d}q \ \frac{P(q)}{q^{n+1}} \ ,$$

  where $\mathcal{C}_\rho$ is a circle of radius $\rho < 1$ centered at the origin in the q-plane.
- $P(q)$ has poles on the unit circle at all primitive roots of unity i..e, at $q = \exp(2\pi h/k)$ for $k = 1, 2, \ldots$ and $h \in [1, \ldots, k-1]$ with $(h, k) = 1$.
- The "strength" of the pole decreases with increasing k and the the contribution to $p(n)$ is largest for $k = 1$. A simple saddle point computation shows that the $k = 1$ term gives

$$p(n) \sim \frac{1}{4\sqrt{3}n} \ e^{\pi\sqrt{\frac{2n}{3}}}$$

  which gives $p(200) \sim 4.10025 \times 10^{12}$.

The Hardy-Ramanujan-Rademacher formula

In 1917, Hardy and Ramanujan computed the contributions from the k-roots of unity and obtained an asymptotic formula. The formula was improved (in 1936) to a convergent one by Rademacher leading to the HRR formula.

$$p(n) \sim \sum_{k=1}^{N(n)} \sum_{\substack{h=1 \\ (h,k)=1}}^{k-1} e^{-2\pi i n h/k} \times \tfrac{2\pi}{k}\left(\tfrac{\pi}{6k}\right)^{3/2} e^{\pi i s(h,k)}$$

$$\times\, d_k(n)^{-3/2} I_{3/2}\left(d_k(n)\right) + O(1)\ ,$$

where $s(h,k) = \sum_{m=1}^{k-1}((\frac{m}{k}))((\frac{mh}{k}))$ is the Dedekind sum and $d_k(n) = \frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}$ and $N(n) \sim n^{1/2}$.
Here is what Hardy had to say: *At this point we might have stopped had it not been for Major MacMahon's love of computation. MacMahon was a practised and enthusiastic table of $p(n)$ up to $n = 200$.*

$p(200)$ using the HRR formula

. . . we naturally took this value as a test for our asymptotic formula. We expected a good result, with an error of perhaps one or two figures, but we had never dared to hope for such a result as we found.[5]

1	$+3,972,998,993,185.896$
2	$+36,282.978$
3	-87.555
4	$+5.147$
5	$+1.424$
6	$+0.071$
7	$+0.000$
8	$+0.043$
$p(200)$	$3,972,999,029,388.004$

The error on adding eight terms was only 0.004! The formula (pre-Rademacher's improvement) was asymptotic and exact.

[5]Table taken from G. Andrews, *Integer Partitions, Cambridge Univ. Press.*

$p(10^{20})$ using the HRR formula

- We saw that the time needed to compute $p(n)$ via the recursion relation grew as $O(n^{3/2})$.
- Fredrik Johannsen showed that it was possible to implement the HRR formula such that the time grew as $O(n^{1/2} \log^{4+o(1)} n)$.
- On March 2, 2014, he announced the computation of $p(10^{20})$ using the HRR formula. He found a 11,140,086,260 digit number! (http://fredrikj.net/blog/2014/03/new-partition-function-record/)

$$18381765083448826436446 \ldots\ldots\ldots 21126231756788091448$$

It took four and a half days on a system with a Xeon E5-2650 CPU and 256 GB of RAM!

A HRR formula for plane partitions

- Building on the work of Almkvist(1998), with Naveen Prabhakar, I obtained an asymptotic formula for plane partitions. [SG-NP, 2013]
- A new function, we call it the *Almkvist* function, appears and can be written in terms of the gen. hypergeom. function ${}_0F_2$.

$$p_2(n) \sim \sum_{k=1}^{[N(n)]} \sum_{\substack{h=1 \\ (h,k)=1}}^{k-1} e^{-2\pi i n h/k} \ \frac{c^k}{k} \left(\frac{a}{k}\right)^{\frac{1}{2}+\frac{k}{24}} e^{C_{h,k}} \times$$

$$\times \sum_{m=0}^{[M^*(n,k)]} b_{h,k}^{(m)} \left(\frac{a}{k^3}\right)^{\frac{m}{2}} \mathcal{A}\left(\left(\frac{a}{k^3}\right)^{\frac{1}{2}} n \,\middle|\, \frac{-k}{12} - m\right) + \mathcal{O}\left(e^{-29.47\frac{n^{1/3}}{k} + 2.01\frac{n^{2/3}}{k}}\right)$$

with $N(n) \sim (2.95 n^{1/3} - 1.47 \log n + 6.4)$, $a = \zeta(3)$, $c = e^{\zeta'(-1)}$, $M^*(n,k) \sim 29.47 n^{1/3}/k$, $C_{h,k} = \frac{k}{2} \sum_{j=1}^{k-1} B_2(j/k) \log|2 \sin(\pi j h/k)|$ is a generalised Dedekind sum.

A HRR formula for plane partitions

The formula is exact for $n < 6425$ and illustrate below for $n = 750$.

1	2545743024358645039521920749024859571789657217789975418420497702709720.300
2	1169353378721087578836884133296412.054
3	1308038187203153215044.287
4	$-$766248063769796.487
5	249747729385.715
6	258376791.876
7	$-$3577528.999
8	$-$1684.466
9	$-$13708.658
10	1766.734
11	$-$274.759
12	$-$61.857
13	$-$6.938
14	0.409
15	2.541
16	$-$0.138
17	$-$0.447
Total	2545743024358645039521920749024859572959010596512371034678586927966061.167
Exact	2545743024358645039521920749024859572959010596512371034678586927966061.000

The current implementation of this formula in mathematica takes about $75s$ for $p_2(6425)$ (on an Intel i5 2.6GHz processor) and is faster than using the following recursive formula:
$p_2(n) = \frac{1}{n}\sum_{k=1}^{n-1} \sigma_2(k) p_2(n-k)$.

Organising partitions by symmetry

Conjugation in higher-dimensional partitions

- The analog of conjugation is now the permutation group S_{d+1} – it permutes the $(d+1)$ axes in the corresponding Ferrers's diagram.
- We can organise the FD's by the action of S_{d+1}.
- For instance, all plane partitions of 4 can be written as

$$\boxed{4 \quad \begin{array}{c}1\\1\\1\\1\end{array} \quad 1\ 1\ 1\ 1} \qquad \boxed{\begin{array}{ll}2 & 1\\1 & \end{array}} \qquad \boxed{\begin{array}{ll}1 & 1\\1 & 1\end{array} \quad 2\ 2 \quad \begin{array}{c}2\\2\end{array}}$$

$$\boxed{3\ 1 \quad \begin{array}{c}3\\1\end{array} \quad 2\ 1\ 1 \quad \begin{array}{c}2\\1\\1\end{array} \quad \begin{array}{lll}1 & 1 & 1\\1 & & \end{array} \quad \begin{array}{ll}1 & 1\\1 & \\1 & \end{array}}$$

- The dim of cosets of S_3/H for $H = S_3, S_2, S_1$ are resp. $1, 3, 6$.
- We call a plane partition, totally symmetric, if it is S_3 invariant. [c.f. Arvind's talk]

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$$\boxed{3\ 1 \qquad \begin{array}{c}3\\1\end{array} \qquad 2\ 1\ 1 \qquad \begin{array}{c}2\\1\\1\end{array} \qquad \begin{array}{lll}1&1&1\\1&&\end{array} \qquad \begin{array}{ll}1&1\\1&\\1&\end{array}}$$

- The dim of cosets of S_3/H for $H = S_3, S_2, S_1$ are resp. $1, 3, 6$.
- Consider all subgroups of S_3, there is one more class corresponding to $\mathcal{C}_3$, the cyclic group. The first non-trivial example of a plane partition invariant under $\mathcal{C}_3$ but not S_3 occurs at $n = 13$. Construct this plane partition!

- An important simplification occurs if we treat partitions in all dimensions on the same footing.
- Consider the FD for the partition of 2 in 1/2/3 dimensions.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

  The nodes lie in the one-dimensional hyperplane $x_i = 0$ for $i = 2, 3, \ldots$. In other words, the only non-zero coordinate is x_1.
- Def: The intrinsic dimension of an FD is defined to be the minimal dimension of hyperplane that contains all its nodes.
- All partitions of 2 have intrinsic dimension $r = 1$.
- A d-dimensional partition with intrinsic dimension (id) r is invariant under S_{d+1-r}. i.e., the permutation of the axes of the orthogonal to the minimal hyperplane of dimension r.

The binomial transform

- What is the number of d-dimensional partitions of 2?

$$p_d(2) = \binom{d+1}{1} \, {\color{red}1} = (d+1).$$

- An FD of id r has the action of S_r – the permutation of the axes of the minimal hyperplane. Let H denote the subgroup of S_r that acts trivially on the FD.
- The number of distinct FD's obtained by the action of S_{d+1} on a FD is given by coset $S_{d+1}/(S_{d+1-r} \times H)$.
- The number of coset elements is

$$\tfrac{(d+1)!}{(d+1-r)! \times \dim(H)} = \binom{d+1}{r} \times \frac{r!}{\dim(H)} := \binom{d+1}{r} \times \text{weight} \ .$$

The binomial transform

- We say that a d-dimensional FD is strict if its intrinsic dimension is $(d+1)$ and not something smaller.
- Now consider, all strict FDs with n nodes and id r. Define

$$\boxed{a_{n,r} := \text{ number of strict FDs with id } r \text{ and } n \text{ nodes }.}$$

- Note that $a_{nr} = 0$ when $r \geq n$. It is a lower-triangular matrix.
- This leads to an interesting formula $p_d(n)$.

$$p_d(n) = \sum_{r=0}^{n-1} \binom{d+1}{r} a_{nr} \;. \quad \text{Binomial Transform}$$

- An example:

$$\begin{aligned} p_d(3) &= \tbinom{d+1}{1} w\left(\ydiagram{3}\right) + \tbinom{d+1}{2} w\left(\ydiagram{2,1}\right) \\ &= \tbinom{d+1}{1}\, 1 + \tbinom{d+1}{2}\, 1 \implies \boxed{a_{3,1} = a_{3,2} = 1}\;. \end{aligned}$$

Properties of the matrix A

- $a_{n0} = \delta_{n,1}$ – this follows since there is precisely one FD with $id = 0$: $\square$. It has $n = 1$.
- $a_{r+1,r} = 1$ for all $n \geq 1$ – again there is only one FD of size $(r+1)$ and id r.
- $a_{n,r} = 0$ when $r \leq n$. It is impossible to construct a FD of id r with fewer than $r+1$ nodes.
- This implies that the $(n-1)$ non-zero numbers in $a_{n,r}$ determine partitions of n in all dimensions. [Atkin et. al. (1967)]

$$A = (a_{n,r}) = \begin{pmatrix}
1 \\
0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 3 & 1 \\
0 & 1 & 5 & 6 & 1 \\
0 & 1 & 9 & 18 & 10 & 1 \\
0 & 1 & 13 & 44 & 49 & 15 & 1 \\
0 & 1 & 20 & 97 & 172 & 110 & 21 & 1 \\
0 & 1 & 28 & 195 & 512 & 550 & 216 & 28 & 1 \\
0 & 1 & 40 & 377 & 1370 & 2195 & 1486 & 385 & 36 & 1 \\
0 & 1 & 54 & 694 & 3396 & 7603 & 7886 & 3514 & 638 & 45 & 1
\end{pmatrix}$$

The above matrix can be directly enumerated using the Bratley-McKay algorithm.

Yet another transform

- Define $F = (f_{n,x})$, as a transform of the matrix A.

$$a_{m+r+1,r} = \sum_{x=0}^{r} \sum_{p=x}^{m} \binom{r}{x} \binom{\binom{r-x}{2}}{m-p} f_{p+x+1,\,x} \; .$$

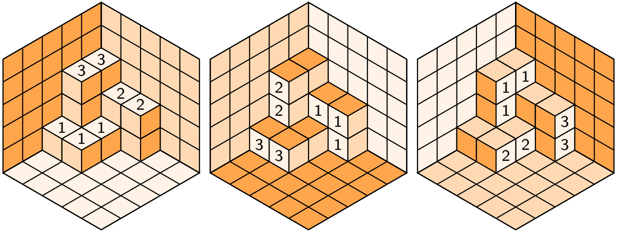
- The main result is that the n-th row of F has only $[(n-1)/2]$ independent numbers. [SG, 2012]
- We illustrate the gain by explicitly displaying the first eleven rows of the A and F-matrices.

$$A = \begin{pmatrix}
1 \\
0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 3 & 1 \\
0 & 1 & 5 & 6 & 1 \\
0 & 1 & 9 & 18 & 10 & 1 \\
0 & 1 & 13 & 44 & 49 & 15 & 1 \\
0 & 1 & 20 & 97 & 172 & 110 & 21 & 1 \\
0 & 1 & 28 & 195 & 512 & 550 & 216 & 28 & 1 \\
0 & 1 & 40 & 377 & 1370 & 2195 & 1486 & 385 & 36 & 1 \\
0 & 1 & 54 & 694 & 3396 & 7603 & 7886 & 3514 & 638 & 45 & 1
\end{pmatrix} , \quad F = \begin{pmatrix}
1 \\
0 \\
0 & 1 \\
0 & 1 \\
0 & 1 & 3 \\
0 & 1 & 7 \\
0 & 1 & 11 & 16 \\
0 & 1 & 18 & 58 \\
0 & 1 & 26 & 135 & 125 \\
0 & 1 & 38 & 293 & 618 \\
0 & 1 & 52 & 574 & 1927 & 1296
\end{pmatrix} .$$

Exercise: Write a program to directly enumerate F!

The smallest plane partition that is cyclically symmetric but not totally symmetric

$$\begin{matrix} 3 & 3 & 1 \\ 2 & 1 & 1 \\ 2 & & \end{matrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$



The plane partition and its $\mathcal{C}_3$ images – it is easy to see that they are indeed the same.

Thank you!

Strongly Recommended: D. Bressoud, *Proofs and Confirmations*.

$$\boxed{p_{100}(26) = 2219401766812531648704448484418 40}$$

http://boltzmann.wikidot.com/the-partitions-project

Can you help this project? Contributions can be in the form of code, algorithms or theoretical insights. If you are an undergraduate student and wish to contribute to this project, send an email to solidpartitions at gmail.com.