Unique Factorization of Symmetric functions

R. Venkatesh

TIFR

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In this talk

- Realization of symmetric groups as reflection groups
- Some examples of symmetric functions
- Relations between symmetric functions
- Unique factorization of Schur functions
- Applications to representation theory (if time permits)

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A few facts from Linear algebra

- V – a finite dimensional vector space
- $\mathcal{B}$ – a basis of V
- $T : V \to V$ be an endomorphism of V
- The action of T is completely determined by its action on $\mathcal{B}$
- $T(\mathcal{B})$ is a basis of V if and only if T – is invertible

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Reflection groups

- Fix $n \in \mathbb{N}$ throughout
- Denote $(\ ,\ )$ by the standard inner product on $\mathbb{R}^n$
- $\mathcal{E}_n = (\mathbb{R}^n, (\ ,\ ))$ Euclidean space
- Given $0 \neq \alpha \in V$, $\alpha^\perp = \mathcal{H}_\alpha := \{\beta \in \mathcal{E}_n | (\beta, \alpha) = 0\}$
- Define $s_\alpha : \mathcal{E}_n \to \mathcal{E}_n$ by

$$s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

- s_α is the reflection with respect to α

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- $s_\alpha(\alpha) = -\alpha$
- $s_\alpha(\beta) = \beta$ for $\beta \in \mathcal{H}_\alpha$
- $[s_\alpha] = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}$ with respect to $\mathcal{B}(\mathcal{H}_\alpha) \cup \{\alpha\}$
- A subgroup of $GL(\mathcal{E}_n)$ generated by finitely many reflections is called reflection group.

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Symmetric group

- $\mathcal{B} = \{\varepsilon_1, \cdots, \varepsilon_n\}$ – the standard basis of $\mathcal{E}_n$
- Given $\sigma \in S_n$, define $\sigma : \mathcal{B} \to \mathcal{B}$ by $\sigma(\varepsilon_i) = \varepsilon_{\sigma(i)}$
- Extend σ linearly to $\mathcal{E}_n$, explicitly

$$\sigma\left(\sum_{i=1}^{n} \lambda_i \varepsilon_i\right) = \sum_{i=1}^{n} \lambda_i \varepsilon_{\sigma(i)}$$

- This gives an action of S_n on $\mathcal{E}_n$:

$$\rho : S_n \to GL(\mathcal{E}_n)$$

- ρ is faithful (one to one map)

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S_n as a reflection group

- Let $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in \mathbb{R}^n$
- Since $(\varepsilon_i - \varepsilon_j, \lambda) = \lambda_i - \lambda_j$, we have $\mathcal{H}_{\varepsilon_i - \varepsilon_j} = \{(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n : \lambda_i = \lambda_j\}$
- Denote s_{ij} by the reflection with respect to $\varepsilon_i - \varepsilon_j$

$$\begin{aligned} s_{ij}(\lambda) &= \lambda - (\lambda_i - \lambda_j)(\varepsilon_i - \varepsilon_j) \\ &= \lambda_1\varepsilon_1 + \cdots + \lambda_j\varepsilon_i + \cdots + \lambda_i\varepsilon_j + \cdots + \lambda_n\varepsilon_n \\ &= \rho_{(i,j)}(\lambda) \end{aligned}$$

- thus $s_{ij} = \rho_{(i,j)}$ and we know that S_n is generated by (i, j)

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Vieta's Formulas

- $X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n = \prod_{i=1}^{n} (X - x_i)$
- $\prod_{i=1}^{n} (X - x_i) = X^n - \left(\sum_{i=1}^{n} x_i \right) X^{n-1} + \cdots + (-1)^n x_1 \cdots x_n$
- $a_m = (-1)^m \sum_{1 \le i_1 < \cdots < i_m \le n} x_{i_1} \cdots x_{i_m}$
- Elementary Symmetric Polynomials:

  $e_m(x_1, \cdots, x_n) = \sum_{1 \le i_1 < \cdots < i_m \le n} x_{i_1} \cdots x_{i_m}$
- $e_0 = 1$, $e_2(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3 + x_1 x_3$

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More Examples

- Powe sums: $p_m(x_1, \cdots, x_n) = x_1^m + \cdots + x_n^m, \quad m \in \mathbb{Z}_+$
- Complete homogeneous symmetric polynomial of degree k in n variables is the sum of all monomials of total degree k in those n variables
- $h_k(x_1, \cdots, x_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$
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Relations between symmetric functions

- $\sum_{r=0}^{m} (-1)^r e_r h_{m-r} = 0$
- $h_k = \det(e_{1-i+j})_{1 \le i,j \le k} = \begin{vmatrix} e_1 & e_2 & \cdots & e_{k-1} & e_k \\ 1 & e_1 & \cdots & e_{k-2} & e_{k-1} \\ 0 & 1 & \cdots & e_{k-3} & e_{k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e_1 & e_2 \\ 0 & 0 & \cdots & 1 & e_1 \end{vmatrix}$
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Relations between symmetric functions

- By symmetry between e and h, we get

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- $p_k = \begin{vmatrix} e_1 & e_2 & \cdots & e_{k-1} & e_k \\ 1 & e_1 & \cdots & e_{k-2} & (k-1)e_{k-1} \\ 0 & 1 & \cdots & e_{k-3} & (k-2)e_{k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e_1 & 2e_2 \\ 0 & 0 & \cdots & 1 & e_1 \end{vmatrix}$
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- Set $P^+ = \sum_{i=1}^{n} \mathbb{Z}_+ \varepsilon_i$.
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Algebra of Symmetric functions in n–variables

- $\mathbb{C}[P^+] = \left\{ \sum_{\alpha \in P^+} a_\alpha x^\alpha : |\text{supp}\{a_\alpha\}| < \infty \right\}$
- $\sum_{\alpha \in P^+} a_\alpha x^\alpha + \sum_{\alpha \in P^+} b_\alpha x^\alpha = \sum_{\alpha \in P^+} (a_\alpha + b_\alpha) x^\alpha$
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Discriminant interms of coefficients

- Fundamental theorem of symmetric functions: Any symmetric polynomial can be expressed as a polynomial in the elementary symmetric polynomials on those variables.
- Recall $X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n = \prod_{i=1}^{n} (X - x_i)$
- $\mathbb{C}[P^+]^{S_n} = \mathbb{C}[e_1, \cdots, e_n]$ where $e_i = e_i(x_1, \cdots, x_n)$
- Discriminant $\Delta = (-1)^{n(n-1)/2} \prod_{i \neq j} (x_i - x_j)$
- If $n = 2$, we have

$$\begin{aligned} \Delta = (x_1 - x_2)^2 &= (x_1^2 + x_2^2 - 2x_1 x_2) \\ &= (x_1 + x_2)^2 - 4x_1 x_2 = a_1^2 - 4a_2 \end{aligned}$$

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- if $n = 3$, we have $\Delta = a_1^2 a_2^2 - 4a_2^3 - 4a_1^3 a_3 - 27a_3^2 + 18a_1 a_2 a_3$
- The discriminant of a general quartic has 16 terms, that of a quintic has 59 terms, that of a 6th degree polynomial has 246
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Schur Functions

- $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in P^+$ is called partition if $\lambda_1 \geq \cdots \geq \lambda_n$
- Gien a partition λ, define $a(\lambda) = \sum\limits_{\sigma \in S_n} (\det\sigma)\, x^{\sigma\lambda}$
- $\delta = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1} = (n-1, n-2, \cdots, 1, 0)$
- Vandermonde determinant:

$$\begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{vmatrix} = \prod_{i<j}(x_i - x_j)$$

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- Gien a partition λ, define $a(\lambda) = \sum_{\sigma\in S_n} (\det\sigma)\, x^{\sigma\lambda}$
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- using Leibniz formula, we get

$$a(\lambda + \delta) = \begin{vmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-2} & \cdots & x_1^{\lambda_{n-1}+1} & x_1^{\lambda_n} \\ x_2^{\lambda_1+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_{n-1}+1} & x_2^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^{\lambda_1+n-1} & x_n^{\lambda_2+n-2} & \cdots & x_n^{\lambda_{n-1}+1} & x_n^{\lambda_n} \end{vmatrix}$$

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Schur Functions

- Given a partition λ, the Schur Function defined to be

$$s_\lambda(x_1, \cdots, x_n) = a(\lambda + \delta)/a(\delta)$$

Theorem (C.S. Rajan, S. Viswanath, –)

Suppose $s_{\lambda_1} \cdots s_{\lambda_p} = s_{\mu_1} \cdots s_{\mu_q}$, then $p = q$ and the multi set $\{\lambda_1, \cdots, \lambda_p\}$ is equal to $\{\mu_1, \cdots, \mu_q\}$. In particular, there exists a permutation $\sigma \in S_p$ such that $s_{\lambda_i} = s_{\mu_{\sigma i}}$.

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Thank you