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Down the rabbit hole of ASMs

1

Background: Systems of lin. eq's

A - $n\times n$ matrix of entries in an arbit field $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

$|A| = \det A$ "determines" whether $Ax = b$ has a sol^n or not.

$$= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

(Leibniz formula)

Eg) $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh - bdi + bfg + cdh - ceg$

where S_n is the set of perms of size n; i.e. the set of bijective maps $\{1, 2, \ldots, n\} \circlearrowleft$

Rep^n: (a) Two-line $\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix} \longrightarrow$ (b) One line $(\sigma(1), \sigma(2), \ldots, \sigma(n))$

$$\mathrm{sgn}(\sigma) = (-1)^{\mathrm{inv}(\sigma)}$$

$\mathrm{inv}(\sigma) = \#$ inversions of σ = # of elem. transp needed to get from σ to $(1, 2, \ldots, n)$

$= |\{i<j : \sigma(i) > \sigma(j)\}|$

(c) Matrices: $\hat{\sigma}$ is a matrix of 0's & 1's assoc to σ.

$$\hat{\sigma}(i,j) = 1 \text{ iff } \sigma(i) = j.$$

Many algorithms
→ Laplace exp
→ Cramer's rule
→ Gaussian elim

Eg) $n=5$, $\sigma = 25314$, $\hat{\sigma} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \sum_{\substack{i>k \\ j<l}} \hat{\sigma}_{i,j} \hat{\sigma}_{k,l}$

$\mathrm{inv}(\sigma) = 3+0+1+1 = 5$

$\Rightarrow$ Magic square where rows & columns sum to 1.

Then
$$|A| = \sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} \prod_{1 \le i,j \le n} a_{i,j}^{\hat{\sigma}(i,j)}$$

$I, J \subset \{1, \ldots, n\}$ $|I| = |J|$

Dodgson Condensation (Lewis Carroll ⟺ C. L. Dodgson) 1866

P. Desnanot - Jacobi (guess 1819) 1833

: Let A^I_J be the matrix A with ith row & jth col removed.

$$\det A = \frac{\det A^1_1 \det A^n_n - \det A^1_n \det A^n_1}{\det A^{1n}_{1n}}$$

where $|A| = 1$ if $n=0$ ~~Possible issue if~~ $|A^{1,n}_{1,n}| = 0$

& $|a| = a$ if $n=1$

Eg) $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \dfrac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} - \begin{vmatrix} b & c \\ e & f \end{vmatrix} \begin{vmatrix} d & e \\ g & h \end{vmatrix}}{|e|}$ which is the same.

Nontrivial : Always a polynomial.

λ-Det : A - $n \times n$ matrix

$$|A|_\lambda := \det_\lambda A = \frac{\det_\lambda A^1_1 \det_\lambda A^n_n + \lambda \det_\lambda A^1_n \det_\lambda A^n_1}{\det_\lambda A^{1,n}_{1,n}}.$$

with $|\cdot|_\lambda = 1$ & $|a|_\lambda = a$.

$|A|_{-1} = A$.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}_\lambda = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix}_\lambda \begin{vmatrix} e & f \\ h & i \end{vmatrix}_\lambda + \lambda \begin{vmatrix} b & c \\ e & f \end{vmatrix}_\lambda \begin{vmatrix} d & e \\ g & h \end{vmatrix}_\lambda}{|e|_\lambda}.$$

$$= aei + \lambda(bdi + afh) + \lambda^2(bfg + cdh)$$

$$+ \lambda^3(ceg) + \underbrace{\lambda(1+\lambda)}_{\lambda^2(1+\lambda^{-1})} bde^{-1}fh$$

$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$0+1+0+0+4$

D. Robbins, H. Rumsey ('86)

~~Thm~~

$$|A|_\lambda = \sum_{A \in \mathcal{A}_n} \lambda^{inv(A)} (1+\lambda^{-1})^{N(A)} \prod M_{i,j}^{A_{ij}}$$

where $\mathcal{A}_n$ set of ASMs of size n.

Def) An ASM is an $n \times n$ matrix of $0, 1, -1$'s s.t.

a) Rows & columns sum to 1.

b) Non-zero entries alternate in each row

Eg $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

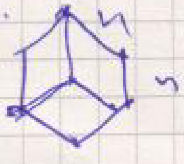
Facts

1) First non-zero entry in every row & col = +1.
2) First row & col has one 1.

$inv(A) \overset{?}{=} \sum_{\substack{i<k \\ j>l}} A_{ij} A_{kl}$, $N(A)$ = # -1's in A.

Q) How many ASMs of size n? $|\mathcal{A}_n| = ?$

- MRR started to ~~count~~ work on this
- Talked to R. Stanley, who told them that this seq was seen before.
- Andrews solved the ~~CSPPs~~ ~~...~~ weak Macdonald conj. by counting # of CSPPs counted there (later)
- He defined DPPs & these came up



* Refined ASMs: Count acc to pos of 1 in 1st row

$A_{n,k}$

k 1

1 2/2 1 1/.

2 2/3 3 3/2 2 1/.

7 2/4 14 5/5 14 4/2 7

42 2/5 105 7/9 135 9/7 105 5/2 42

429 2/6 1287 8/4 2002 16/16 2002 14/9 1287 6/2 429

Num → (2,1) Pascal's $\triangle$le

Denom → (1,2) " "

Data

Prime factors	n	$\lvert\mathcal{A}_n\rvert$
	1	1
2^1	2	2
7^1	3	7 (6+1)
$2\cdot3\cdot7$	4	42 (24+16+2)
$3\cdot11\cdot13$	5	429
$2^2\cdot11\cdot13^2$	6	7436
$2^2\cdot13^2\cdot17\cdot19$	7	218348
$2^3\cdot13\cdot17^2\cdot19^2$	8	10850216
$2^2\cdot5\cdot17^2\cdot19^3\cdot23$	9	911835460

Round

ASM Conjectures (1986) MRR

② $\dfrac{A_{n,k+1}}{A_{n,k}} = \dfrac{(n-k)(n+k-1)}{k(2n-k-1)}$

① $A_n = \prod_{j=0}^{n-1} \dfrac{(3j+1)!}{(n+j)!}$

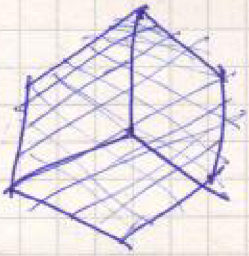
$A_{n,k} = A_{n-1} \dfrac{\binom{n+k-2}{n-1}\binom{2n-k-1}{n-1}}{\binom{2n-2}{n-1}}$

NOTE: I could only cover the material up to here.

DPPs ~~work~~ Macdonald's conj. ('79) ④

Andrews proved #CSPPs :

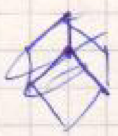
~~#Lozenge tilings of an $n\times n\times n$ hexagon invariant under 120° rot^n. in a~~ ~~Δlat~~ ~~lattice~~



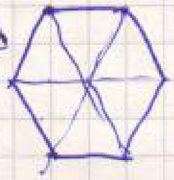
Eg)

6	5	5	4	3	3
6	4	3	3	1	0
6	4	3	1	1	0
4	2	2	1	0	0
3	1	1	0	0	0
1	1	1	0	0	0

CSPP of 75



Handout 1



A DPP of order n is a 2D array of positive integers (a_{ij}) defined for $j \geq i$ ~~that has~~ ~~whose rows & columns are weakly decreasing~~ written in the form

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1,\mu_1} \\ & a_{22} & a_{23} & \cdots & a_{2,\mu_2} \\ & & \ddots & & \\ & & a_{rr} & \cdots & a_{r,\mu_r} \end{array}$$

s.t.

7 7 6 6 3 1
6 5 4 2
3 3
2

① $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r$

② $a_{i,j} \geq a_{i,j+1}$ & $a_{i,j} > a_{i+1,j}$.

③ Let $\lambda_i = \mu_i - i + 1$ = # elements at row i.

$a_{ii} > \lambda_i$ & $a_{i,i} \leq \lambda_{i-1}$

DPPs of order 3 : $\emptyset$, ~~1~~, 2, 3, 31, 32, 33, 33 *, 33 / 2

$\mathcal{D}_n :=$ set of DPPs of order n

$\boxed{a_{ij} \leq j-i}$

MRR 1983 | $|\mathcal{D}_n| = A_n$ (Conj.)

But No known bijection

More refinements

ASM	DPP
– 1 in first row in pos k	$k-1$ parts equal to n
– #−1's = m	m special parts
– Inv# = p	p total parts

TSSCPPs (Handout 2)

Count # plane partitions under all possible symmetries

- All 3 reflections
- Rotation
- ~~Self~~-complementary

$\boxed{\text{Handout \# 2}}$

Let $\mathcal{T}(n)$ = set of TSSCPPs inside a $2n \times 2n \times 2n$ box.

MRR 1986 | Conj | $|\mathcal{T}(n)| = A_n$.

Fundamental region of TSSCPP can be encoded in terms of a Δle s.t. m_{ij} $\quad 1 \le i \le n$, $1 \le j \le i$

- $m_{ij} \le m_{i+1,j}, m_{i,j+1}$
- $m_{ij} \ge j$.

Eg)

$$\begin{array}{ccccc} 1 \\ 2 & 3 \\ 4 & 5 & 5 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{array}$$

Magog Δles

Given ASM, one can also form a Δle as follows

$$\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} & & & & 1 & & \\ & & 1 & & & & \\ & 1 & -1 & 1 & -1 & 1 & \\ & & 1 & -1 & 1 & & \\ & & & 1 & & -1 & 1 \\ 1 & -1 & & & & 1 & \\ & 1 & & & & & \end{pmatrix} \end{array} \longrightarrow \begin{array}{ccccccc} 5 \\ 3 & 5 \\ 2 & 4 & 6 \\ 2 & 3 & 5 & 6 \\ 2 & 3 & 4 & 5 & 7 \\ 1 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

(g_{ij}) satisfies

$g_{ij} \le g_{i-1,j} \le g_{i,j+1}$

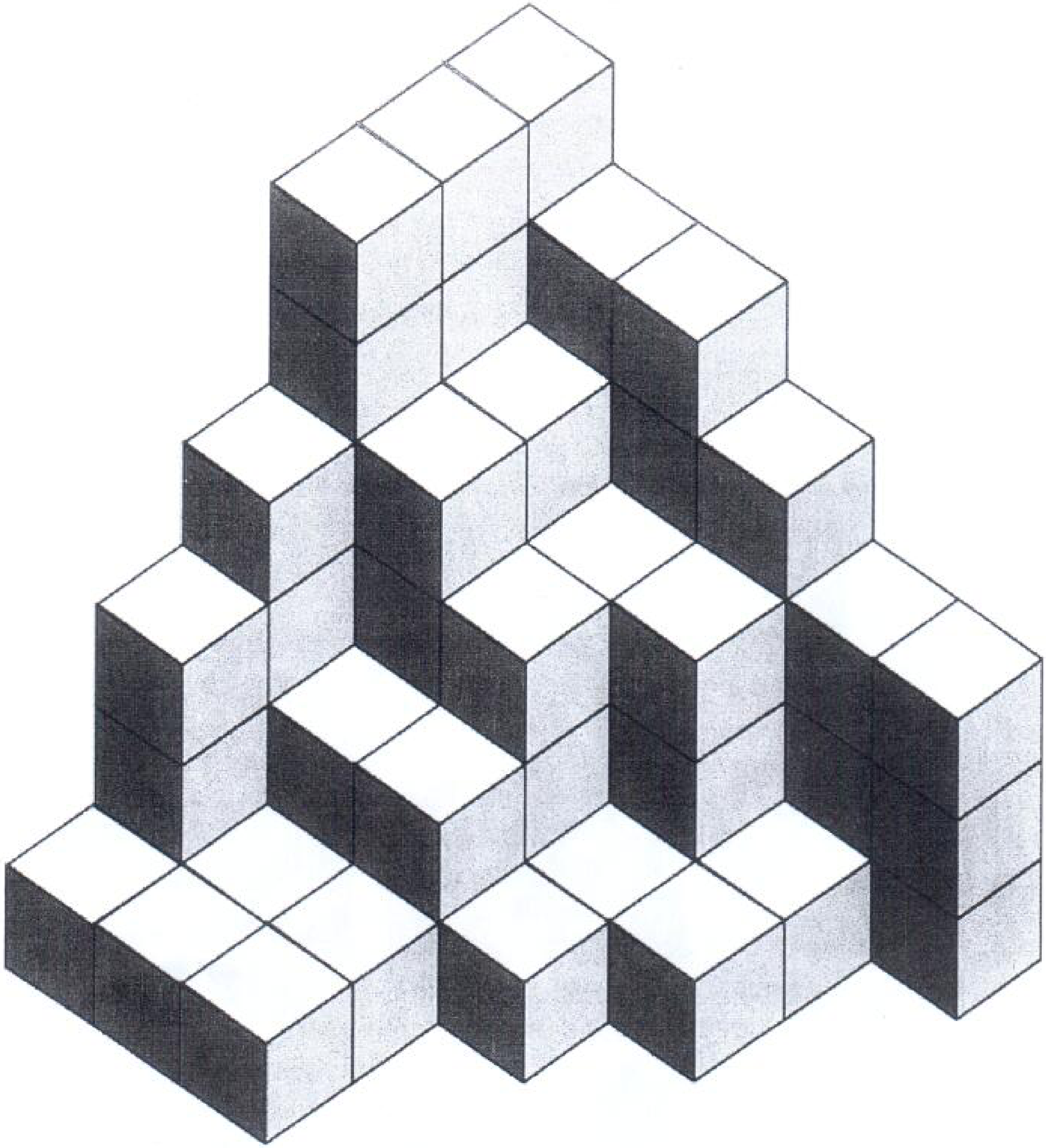
& $g_{ij} < g_{i,j+1}$

Monotone Δles = gogs

~~1992~~ '92 – Zeilberger proved gogs = Magogs in a – page paper with 80 referees

- Kuperberg proved ASM conj in 6 page paper using 6V model. (IK det)
- Zeilberger uses it to prove RASM conj.

Refinements?



An example of a CSPP in
a 6X6X6 box

All 7 TSSCPPs in a
6 X 6 X 6 box