Can Maps Make the World Go Round?

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CMI

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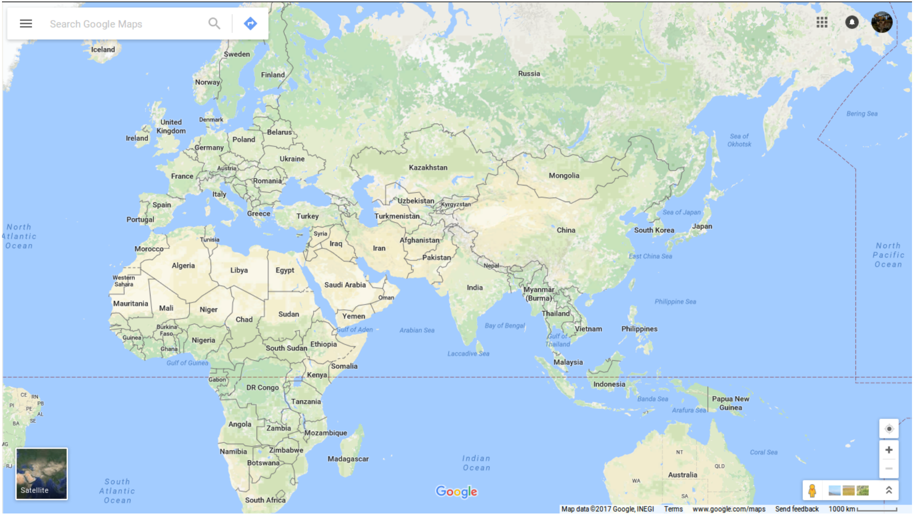
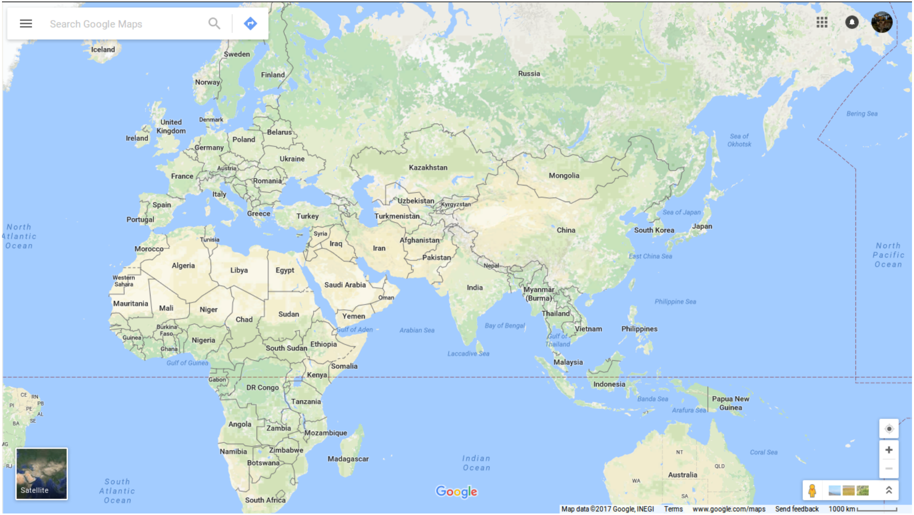
A common map of the Earth.

Maybe you've seen this map before. It's called the Mercator projection.



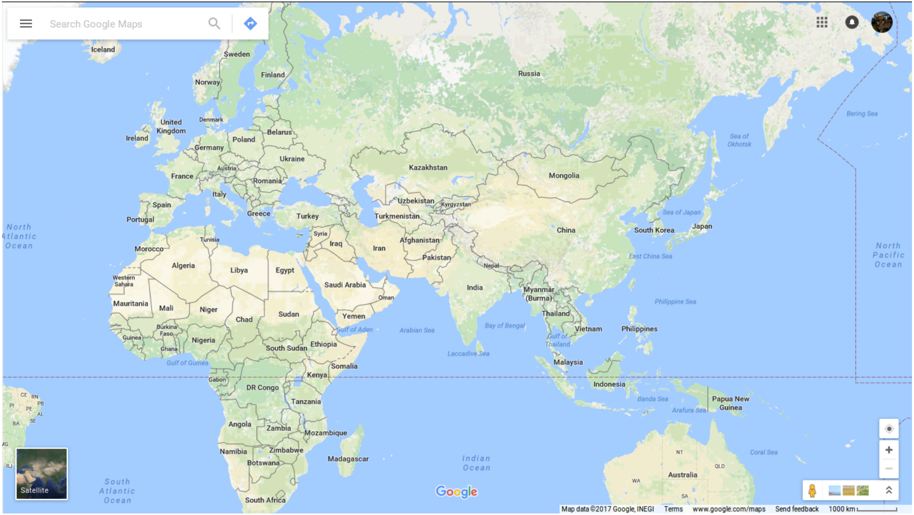
A common map of the Earth.

It is famously imperfect. Can you find any problems with it?



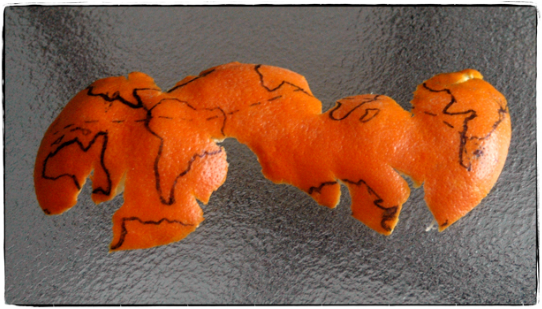
A common map of the Earth.

Given all these problems, why do we use it? Is there a better map?



Is no map perfect?

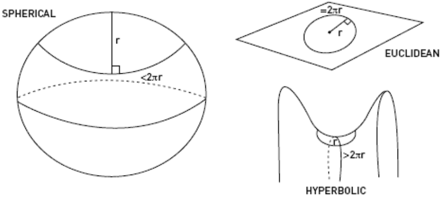
Perhaps you've noticed that it's very hard to completely flatten an orange peel.



This is because all flat images of (even small) portions of the sphere will have distortions.
The reason has to do with **curvature**.

Curvature: An Intuitive Idea

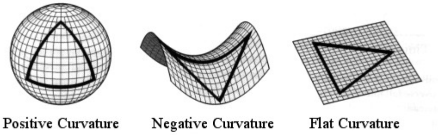
Consider a flat (zero-curvature) surface. The ratio of circumference to radius of any circle is exactly 2π.



However, when curvature around a point P is positive, the circle of radius 1 centered at P will have circumference less than 2π.
And negative curvature will result in a circumference greater than 2π.

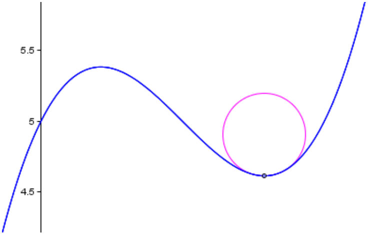
Curvature: An Intuitive Idea

Curvature also affects the angle sum of triangles:



Gaussian curvature of surfaces

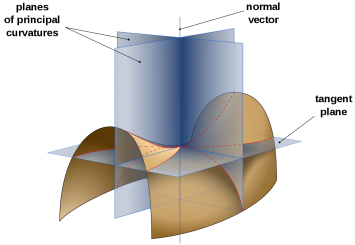
But let's give a more precise definition.



The *radius of curvature* ρ of a plane curve $\mathcal{C}$ at a point P is the radius of the circle that best approximates $\mathcal{C}$ at P.
The *curvature* of $\mathcal{C}$ at P is $\frac{1}{\rho}$.

Gaussian curvature of surfaces

Now let's consider all the plane curves obtained by cutting a surface $\mathcal{S}$ by normal planes at a point P.



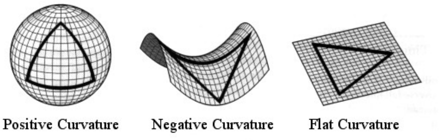
Let ρ_1 be the maximum and ρ_2 be the minimum *signed* radii of curvature at P, where the sign tells us on which side of the surface the circle sits.

The *Gaussian curvature* κ of $\mathcal{S}$ at P is then defined by

$$\kappa = \frac{1}{\rho_1 \rho_2}.$$

Surfaces of constant curvature

The following surfaces all have constant Gaussian curvature (it is the same at any point).



Can you determine the Gaussian curvature of the unit sphere?

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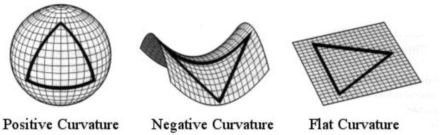
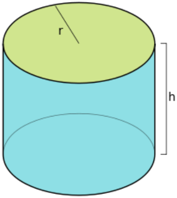


Can you determine the Gaussian curvature of the unit sphere?
How about the cylinder of radius 1 and height h?



A Theorem of Gauss

Theorem (Theorema Egregium, Carl Friedrich Gauss, 1827)

Any **distance-preserving** *map from a surface $S \subset \mathbb{R}^3$ to a surface $R \subset \mathbb{R}^3$ must preserve* **Gaussian curvature**.

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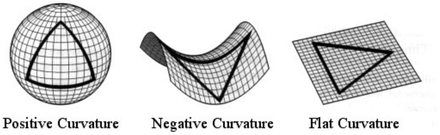
A consequence of Gauss's Theorem



The sphere and the plane have different Gaussian-curvatures at every point.
So any map from the sphere to the plane must distort distance **everywhere**!

So maps will never get distance correctly, but...

There are other properties maps can capture perfectly.

Surprisingly, it is possible to preserve curves of shortest distance, also known as **geodesics**.

Geodesics

What do geodesics on the plane look like? (What is the shortest path between two points on the plane?)

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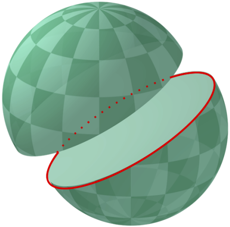
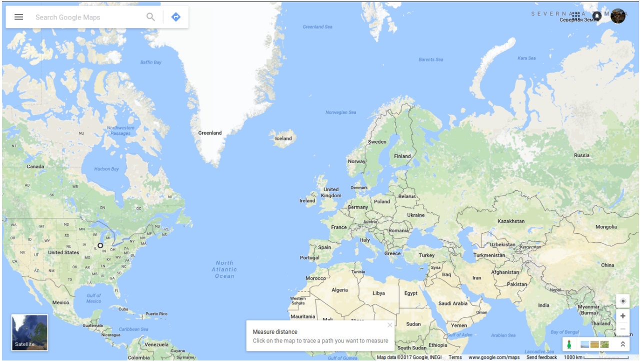


Geodesics on the Mercator projection

The Mercator does not take geodesics to geodesics. For example, if you wanted to fly from Chennai to Chicago quickly, you'd want your path to look like this:



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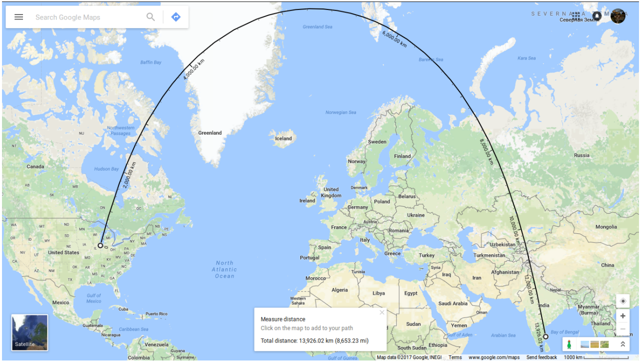
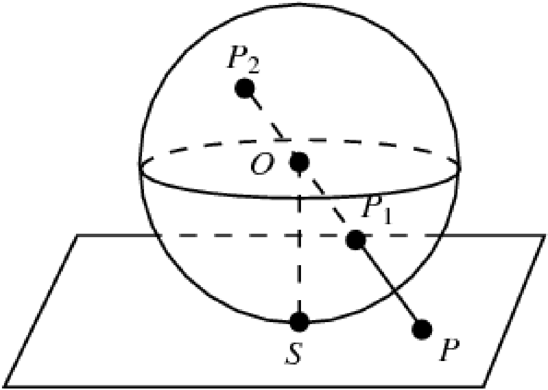


The oldest known projection of the Earth is the **Gnomonic projection**. It takes one hemisphere of the Earth to the entire plane $\mathbb{R}^2$.



A map that preserves geodesics

Here is a portion of a gnomonic projection taken at the North Pole:

Exercise

Prove that the Gnomonic projection takes geodesics to geodesics.

A map that preserves geodesics

Here is a portion of a gnomonic projection taken at the North Pole:

Unfortunately, the Gnomonic projection is not very useful for most applications.
And most maps we commonly use in fact fail to preserve geodesics...

Area

However, it *is* possible to make a projection that preserves area perfectly.

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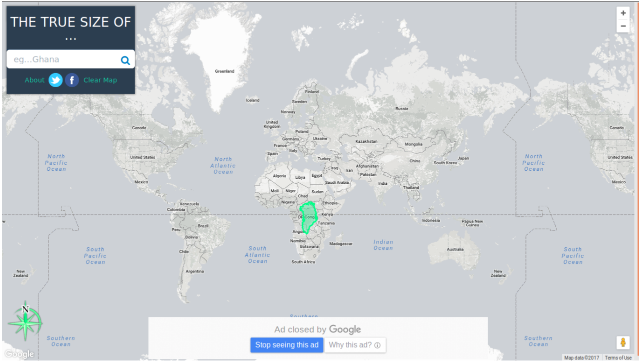
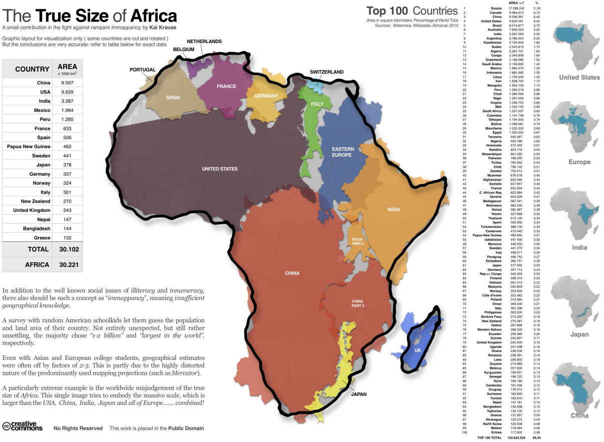
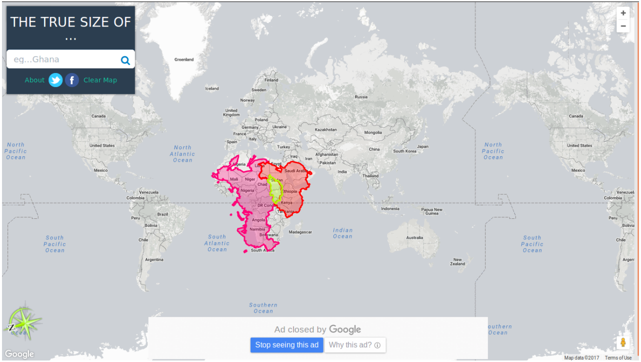


In fact, Africa is much bigger than most people realize, and the 'Western world' is much smaller!



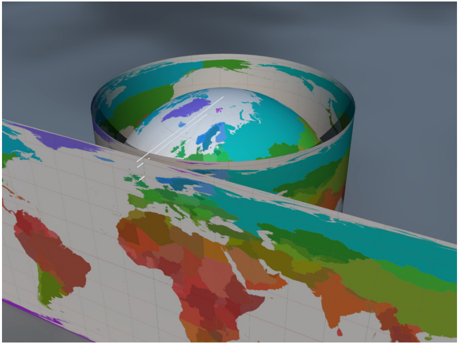
thetruesize.com

You can compare the true sizes of countries at the site thetruesize.com



A map that preserves area

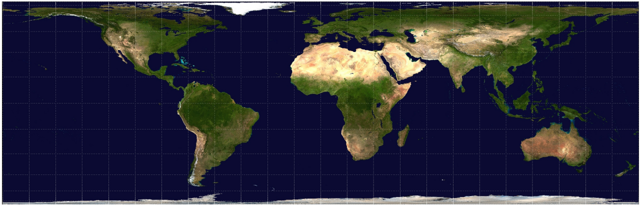
However, there *ARE* maps that perfectly preserve area.
The most elegant example is the cylindrical projection, which maps $S^2 \setminus \{N, S\}$ to the cylinder of the same height and radius.



Note that this projection fixes the z-value (i.e. the height) of all points.

A map that preserves area

When we unwrap the cylinder, we get the following equal-area map:

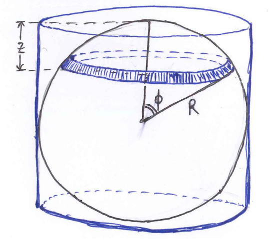


Theorem (Archimedes)

The cylindrical projection preserves area everywhere.

Proof of Archimedes' Theorem

Consider a sphere of radius R, enclosed by a cylinder of radius R and height $2R$.



The key ingredient is the fact that any horizontal slice of the sphere of width dz has surface area equal to $2\pi R dz$.

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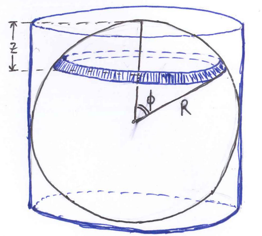
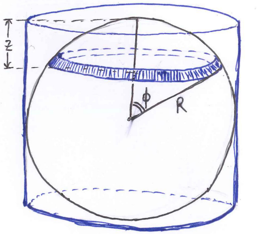


The key ingredient is the fact that any horizontal slice of the sphere of width dz has surface area equal to $2\pi R dz$.
This is also the surface area of the corresponding slice of the cylinder of radius R!

Proof of Archimedes' Theorem

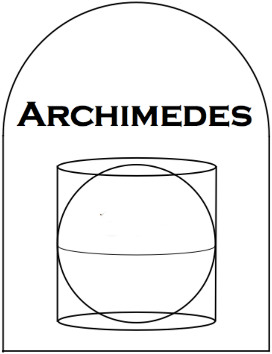
Consider a sphere of radius R, enclosed by a cylinder of radius R and height $2R$.



To see this, note that the width of the strip is $Rd\phi$.
For small dz, we can approximate the surface area of the strip by a frustrum of a cone with radius $R\sin\phi$ and slant height $Rd\phi$.
This frustrum has area $\pi(2R\sin\phi)(Rd\phi)$.
But this quantity is equal to $2\pi Rdz$, since
$dz = d(R - R\cos\phi) = R\sin\phi d\phi$.

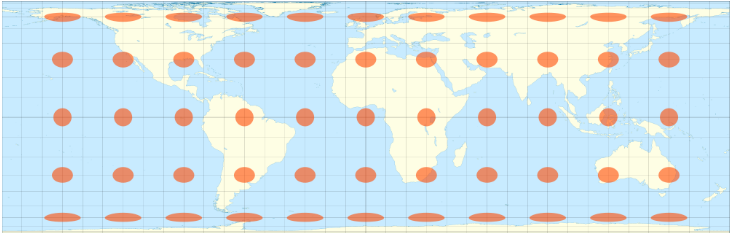
A map that preserves area

Archimedes was so proud of this discovery that he had it engraved on his tombstone!



A map that preserves area

We can see that the cylindrical projection is indeed area preserving by using *Tissot's Indicatrix*:



But we also see that it distorts **shape** quite badly!

What is Shape?

A final property we will consider is **shape**.

More precisely, we will look at maps that preserve angles everywhere.

Such maps are called **conformal** and they preserve the basic shape of objects (with some amount of stretching permitted).

Preserving shape

For example, the map $(u, v) \mapsto (\lambda u, \lambda v)$ is conformal,

Preserving shape

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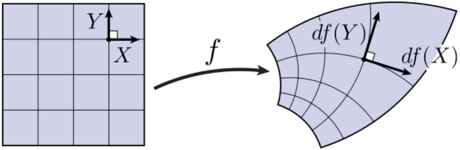
but the map $(u, v) \mapsto (\lambda u, v)$ is not conformal.

Can you say why it is not?

Definition of conformal map

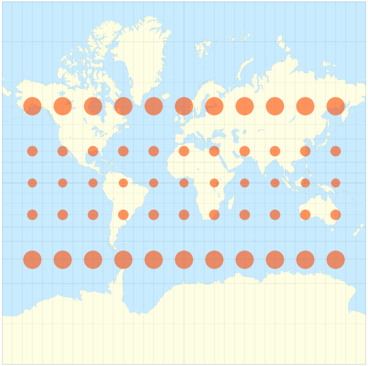
Definition

A map of surfaces $f : S \to R$ is *conformal* if for any point $P \in S$ and any vectors X and Y in the tangent plane $T_P(S)$, the signed angle from X to Y is equal to the signed angle from $df(X)$ to $df(Y)$ in the tangent space $T_{f(P)}(R)$.



Shape in the Mercator projection

Shape is something the Mercator projection gets right.

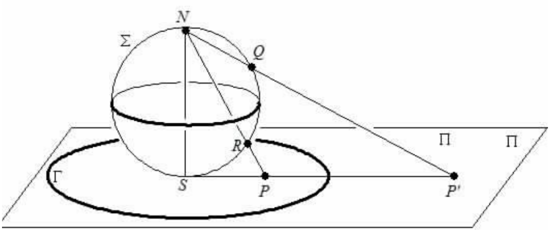


But there is a much simpler map that also preserves shape.

A map that preserves shape

The stereographic projection Φ takes $S^2 \setminus \{N\}$ onto the entire plane $\mathbb{R}^2$.

$$\Phi : (x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \tag{1}$$



To understand this projection, imagine the unit sphere S^2 centered at $(0, 0, 1)$, so that it sits on the xy plane with its south pole S on the origin $(0, 0, 0)$.

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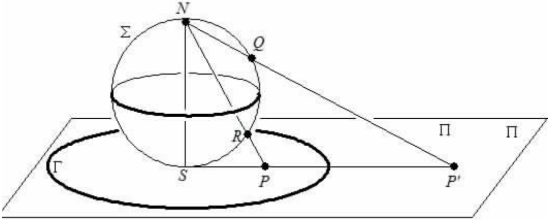
$$\Phi : (x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \tag{2}$$



Now imagine a laser beam that fires from the north pole N, and takes each point of $S^2 \setminus \{N\}$ to a point on the plane.
Can you convince yourself that this projection maps onto the entire xy plane?

A map that preserves shape

Theorem

The stereographic projection $\Phi : S^2 \setminus \{N\} \to \mathbb{R}^2$ *defined by*

$$(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \tag{3}$$

is conformal.

Proof

The proof will require the following three exercises, which you can try to solve on your own.

Exercise

Any rotation of S^2 is the product of two reflections through great circles in S^2.

Exercise

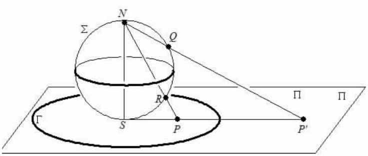
If $\mathrm{ref}_{\mathcal{C}}$ is a reflection through a great circle in S^2, then $\Phi \circ \mathrm{ref}_{\mathcal{C}} \circ \Phi^{-1}$ is an inversion through a circle in $\mathbb{R}^2$.

Exercise

Any inversion through a circle in $\mathbb{R}^2$ preserves magnitudes of angles. (Hint: first show this for the inversion $(u, v) \mapsto \frac{(u,v)}{\|(u,v)\|}$.)

Proof

Using these exercises, the proof becomes much easier!
First convince yourself that Φ preserves angles at the south pole S.



We want to show that it preserves angles at an arbitrary point $Q \neq S$. Let $\text{rot}_Q : S^2 \to S^2$ be the rotation that takes Q to S.
By the first exercise we have $\text{rot}_Q = \text{ref}_{\mathcal{C}_2} \circ \text{ref}_{\mathcal{C}_1}$ for some reflections through great circles $\mathcal{C}_1$ and $\mathcal{C}_2$.
We can write
$\Phi = (\Phi \circ \text{rot}_Q^{-1} \circ \Phi^{-1}) \circ \Phi \circ \text{rot}_Q = (\Phi \circ \text{ref}_{\mathcal{C}_1} \circ \Phi^{-1}) \circ (\Phi \circ \text{ref}_{\mathcal{C}_2} \circ \Phi^{-1}) \circ \Phi \circ \text{rot}_Q$.
By the second and third exercises, each of these maps preserves magnitudes of angles. Therefore the composition does too!

A map that preserves shape and area?

We have seen a map that preserves area and a map that preserves shape. Is it possible for a map to preserve both?

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After taking a course in Differential Geometry, you will be able to prove the following yourself!

Theorem

If a map of surfaces $f : S \to R$ is both conformal and area-preserving, then f is distance-preserving.

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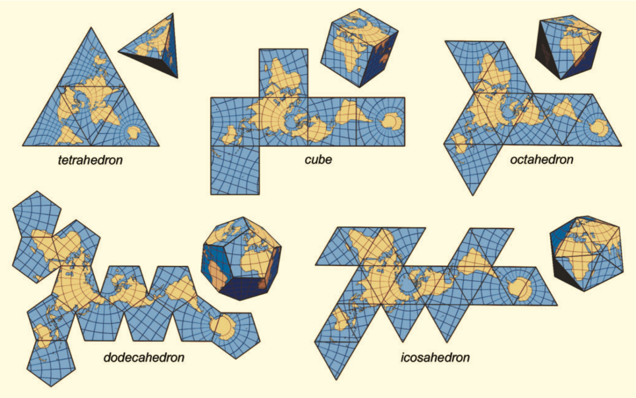
So is it possible for a projection from a part of the sphere S^2 to $\mathbb{R}^2$ to be both area and angle preserving?

Compromises

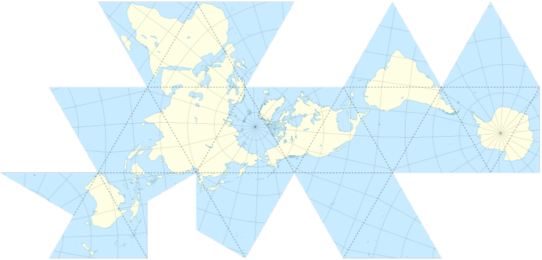
Although it is impossible for a projection to preserve both area and shape, there are projections that do a good enough job of both, especially around areas we care about (like landmasses).

Compromises

Some successful compromise maps have taken an interesting approach: Project the sphere onto an inscribed polyhedron (such as a platonic solid), and unfold:

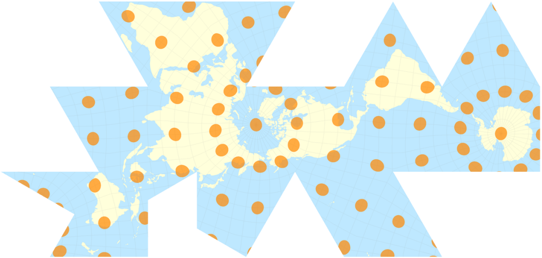


One famous example is the Dymaxion map, made by Buckminster Fuller in 1943. It projects the sphere onto the surface of a regular icosohedron inscribed within.



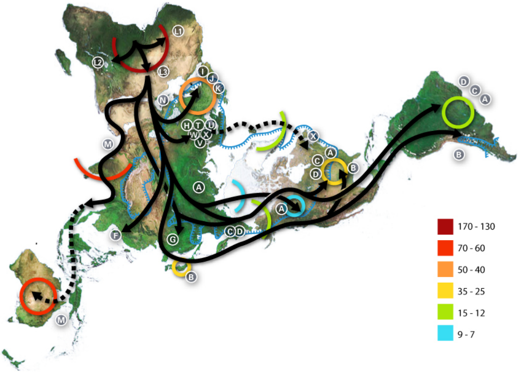
Compromises

The Tissot Indicatrix shows that it indeed does a good job of approximating both shape and area:



Moreover, the Dymaxion map is free of many cultural biases: there is no up or down, no north or south.
In fact, one way of unfolding the map shows that the continents can be viewed as one almost contingous land-mass.

One could even argue that this is a more natural way of viewing the world, since it agrees with human migration routes out of Africa!



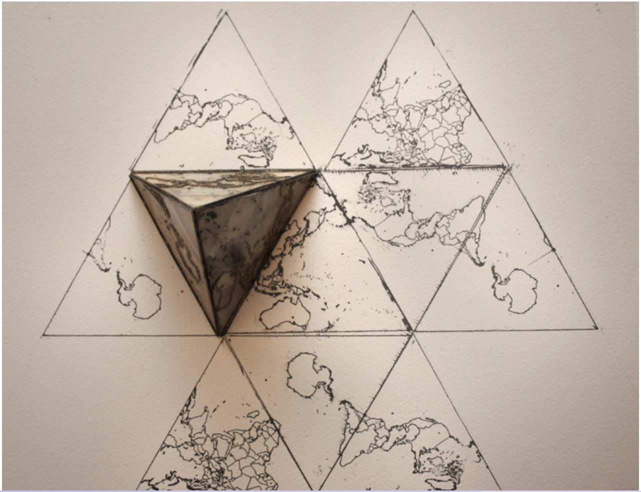
Compromises

The Dymaxion map inspired the Japanese architect Hajime Narukawa to invent the Authugraph map in 1999.

Instead of an icosohedron, the Authagraph projects the sphere onto an inscribed tetrahedron, and uses a special projection to keep the area and shape accurate around the continents.

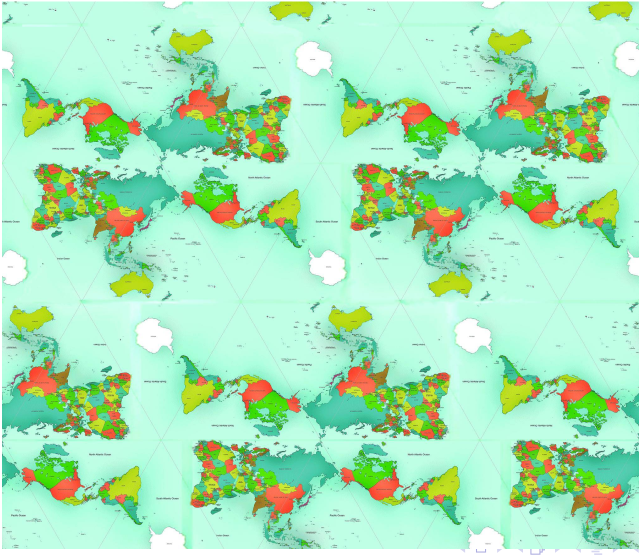
Compromises

The advantage of using a tetrahedron is that the resulting unfolded map is an equilateral triangle. Amazingly, these triangles can be used to tile the plane, resulting in an infinitely repeating center-less map:



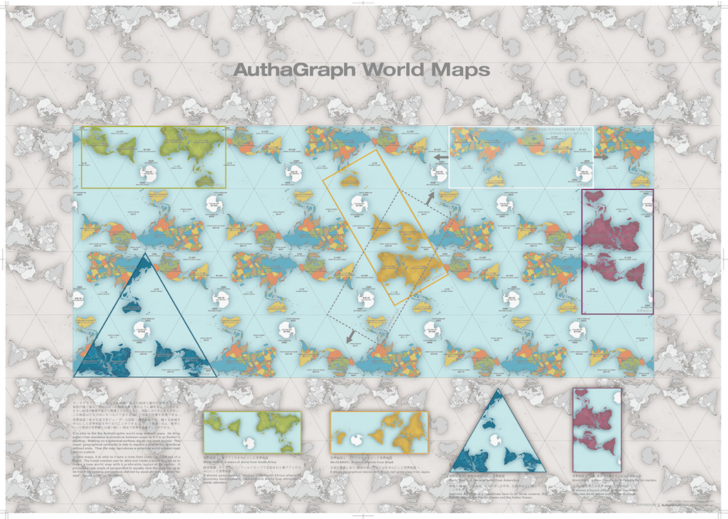
Compromises

This infinite tiling is an example of a *tesselation*:



Compromises

There are many finite maps of Earth one can carve out of this tessellation. For example:



You have a Authagraph map in your folder. We will pass out scissors for you to cut it out.

Try to assemble a a large tiling with the people around you.
See which rectangular maps you can identify in the tiling.
Later, you can fold your map into a tetrahedral globe.