From individual preferences to society's

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- Do feel free to interrupt with questions any time.

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- Transitivity: If you prefer Mac over Acer and Acer over Lenovo, it is reasonable to expect that you prefer Mac over Lenovo.
- Usually preference is not total: some choices are incomparable. But to keep things simple, I will assume totality of preference relations.

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- If 3 prefer Acer over Lenovo and 2 prefer the other way, what can you say about group preference ?

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- So the majority rule is not transitive.

Typical rules

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- Dictatorial: group preference is determined by a specific individual's preferences.
- Choose the "most common" preference in the group.
- Minimize dissatisfaction: that is, minimize the number of individuals whose preference is opposite to what you decide for the group.

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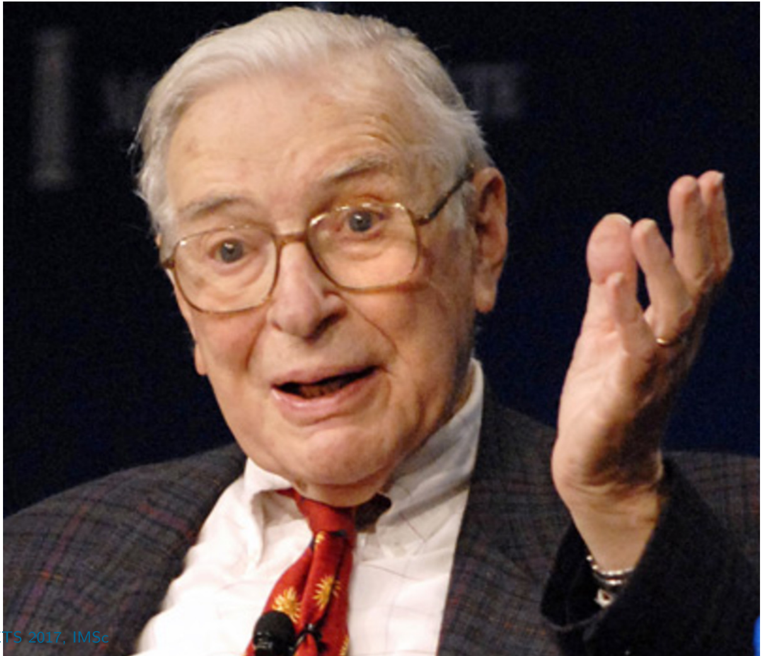
- IIA: If two profiles agree on a pair of choices, then the group preferences derived from them also agree on that pair.
- Non-dictatorial: There is no single individual whose preference unilaterally determines the group preference.

Kenneth Arrow



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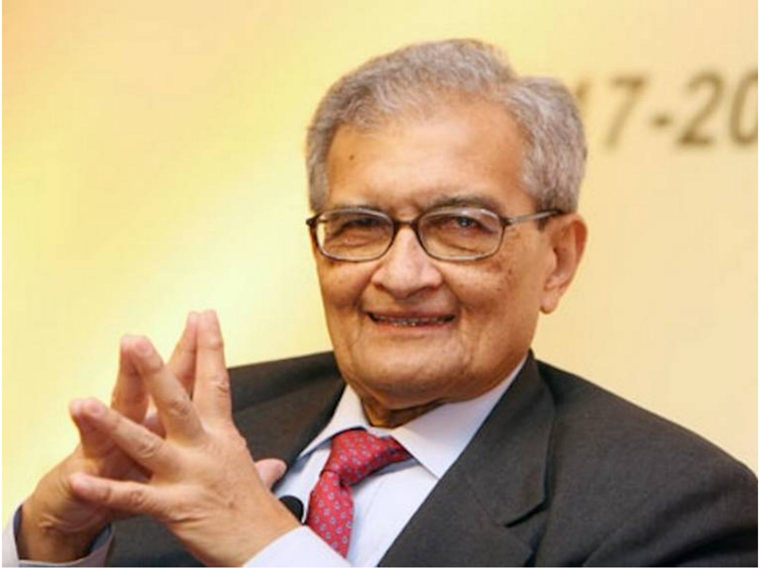
- A beautiful introduction to SCT: Amartya Sen (1970).

Amartya Sen



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- G is almost decisive if whenever everyone in G prefer a over b and everyone outside the group prefer b over a, then the outcome prefers a over b.
- Clearly, every decisive group is almost decisive; we will show that the converse is true as well.

Contraction lemma

Partition G into non-empty G_1, G_2 such that one of the two is almost decisive.

- Fix choices a, b, c and a profile such that over G_1 we have vector cab, over G_2 it is abc and outside G it is bca.

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- Suppose G_1 is not almost decisive; by IIA, we can argue that the outcome prefers b over c.
- Suppose G_2 is not almost decisive; again by IIA, we can argue that the outcome prefers c over a.
- By transitivity, the outcome prefers b over a. But G is decisive and everyone in G prefers a over b, so the outcome prefers a over b as well, a contradiction.

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- Let c be a third alternative. Define R' by: over a, b do the same as R; over G, use $acbx$ for other x; outside G use cax and cbx for other x.

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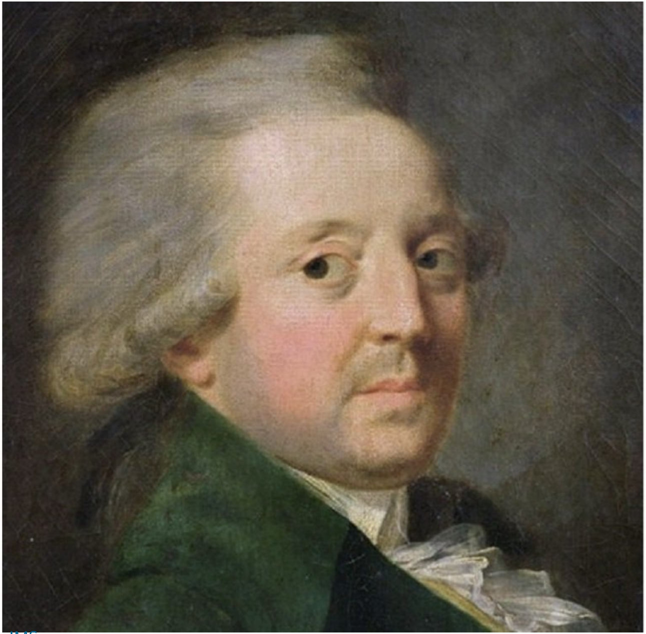
- Let c be a third alternative. Define R' by: over a, b do the same as R; over G, use $acbx$ for other x; outside G use cax and cbx for other x.
- Since G is almost decisive, a is preferred over c. By Pareto, c is preferred over b, and hence a is preferred over b, by transitivity.

Elections

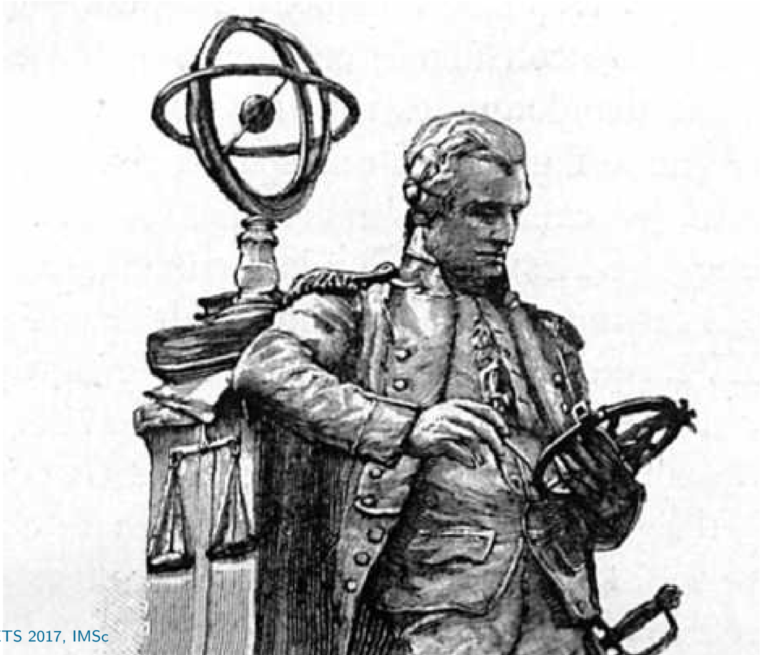
Elections are also about aggregating social preferences from individual preferences.

- 18^{th} century: Condorcet and Borda.
- 19^{th} century: Charles Dodgson. (familiar ?)
- 20^{th} century: Kenneth Arrow, Satterthwaite, ...

Marquis de Condorcet



Jean-Charles de Borda



Voting rules

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- Veto: For each alternative count last place votes, and the ones with the least of them win.

Multi-round rules

Some elections have multiple rounds.

- Single Transferable Vote (STV): There are $m - 1$ rounds. In each round, the alternative with the least plurality votes is eliminated, and the survivor to the last becomes the winner. This is used in Ireland, Malta, Australia, and New Zealand.

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- Plurality with Runoff: In the first round, two alternatives with highest plurality votes survive. In the second round, the winner of a pairwise election between those two alternatives becomes the final winner. The French elections follow this rule.
- Condorcet winner: Conduct pair-wise elections and the winner is one who beats every other alternative in a pair-wise election. Note that a Condorcet winner may not exist, in general.

Condorcet Consistency

A rule is Condorcet consistent if it elects the Condorcet Winner if one exists. Here are some CC rules.

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- Dodgson: A distance function between preference profiles is defined as the number of swaps between adjacent candidates, and the Dodgson score of alternative x is the minimum distance from a profile in which x is a Condorcet Winner.

Computing winners

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- Computing Dodgson winner is NP-complete.
- Many restrictions on preference profiles are being studied.

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We say that a rule is manipulable if there exists a profile where a voter i can switch her preference from R_i to R_i' such that her most preferred (in R_i) wins.

- Theorem (Gibbard – Satterthwaite): If a voting rule has at least 3 possible outcomes and is non-manipulable, then it is dictatorial.

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- Complexity comes to the rescue: in many systems, manipulation is NP-hard.

A rich field

An area of intellectual endeavour that intersects philosophy, politics and economics, and uses tools from mathematics and computer science.

- Societies cannot be rigid about voting rules.
- Social preferences are hard to derive and the logical difficulties in doing so need to be acknowledged and addressed.
- With the advent of the internet and algorithms that make decisions, these considerations apply to a far wider variety of contexts than before.

Discussion time

Thank you.

Questions, comments, suggestions welcome; also, please write to jam@imsc.res.in.