Some Algorithmic questions in Finite Group Theory

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Plan of Talk

- Permutation groups: background and some basic algorithms.

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- Random subproducts in groups.

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• What's new is probably the cocktail of group theory + algorithms.

• Teachers: Experimenting with similar cocktails in other courses like linear algebra can be interesting.

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- We can "describe" a permutation group G by listing down all its elements. A more compact description is to give a generating set for it.

- The permutation group $\langle S \rangle$, *generated* by a subset $S \subseteq S_n$ of permutations, is the smallest subgroup of S_n containing S.

Permutation Groups: Definitions Contd.

• Every finite group G has a generating set of size $\log_2 |G|$.
Because

$$\langle g_1 \rangle < \langle g_1, g_2 \rangle < \ldots < \langle g_1, g_2, \ldots, g_k \rangle = G.$$

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So, giving a generating set for G is a succinct representation as it as algorithmic input.

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• The group G is called *transitive* if there is exactly one orbit.

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• What is an efficient algorithm on permutation groups?

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• Roughly speaking, for algorithm dealing with permutation groups in S_n:

Polynomial in n many operations = Efficient.

Exponential in n operations = Inefficient.

Computing Orbits Efficiently

- Given $G \leq S_n$ by a generating set S, we can compute the orbit of any point i in $(n|S|)^{O(1)}$ time.

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$O := \{i\}$;
while O changes **do**
$O := O \cup \{i^{g_j} \mid i \in S, 1 \leq j \leq k\}$;
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• The loop runs for at most n steps. In the loop the number of operations is bounded by $O(nk)$. Thus, $O(n^2 k)$ operations in all.

The Membership Testing Problem

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Example Consider the cyclic group $G = \langle g \rangle$, where g is a permutation of order $2^{O(\sqrt{n \log n})}$ (by choosing g to be a product of cycles of prime length for different primes).

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Example Consider the cyclic group $G = \langle g \rangle$, where g is a permutation of order $2^{O(\sqrt{n \log n})}$ (by choosing g to be a product of cycles of prime length for different primes).

We need to have a more compact way of expressing $\pi \in G$ in terms of its generators.

Membership Testing Contd.

Elements of G are g^b where b is $t = O(\sqrt{n \log n})$ bits. We compute g^b by repeated squaring and multiplying the appropriate powers g^{2^i}. Let $b = \sum_{i=0}^{t-1} b_i 2^i$.

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The following *straight-line program* computes g^b:

$x_0 := g$;
for $i := 1$ to $t-1$ **do**
$x_i := x_{i-1}^2$;
$x_t := 1$;
for $i := 1$ to $t-1$ **do**
$x_{t+i} := x_{t+i-1} \cdot x_i^{b_i}$;

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• **Nice Fact** If $\pi \in \langle g_1, g_2, \ldots, g_k \rangle \leq S_n$ then it has a straight-line program of length polynomial in n and k, and the membership testing algorithm will find in $poly(n)$ time.

Strong Generating Sets

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- "Strong" because every $\pi \in G$ can be expressed uniquely as a "short" product $\pi = \pi_{n-1}\pi_{n-2}\ldots\pi_1$, where $\pi_i \in T_i$.

Back to Membership Testing

• Given a strong generating set for $G \leq S_n$, membership testing is easy and efficient. Let $\pi \in S_n$:

$$\pi \in G \iff \pi \in G_{[1]}\pi_1 \text{ for } \pi_1 \in T_1.$$

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• How do we find a strong generating set for G?

Finding a Strong Generating Set

• Given $G = \langle S \rangle$, finding T_1 is easy. We can compute the orbit O of 1, and for each $j \in O$ keep track of a $\pi_1 \in G$ such that $1^{\pi_1} = j$. How do we find T_2?

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Schreier's Lemma Let $G = \langle A \rangle$ and $H \leq G$ of finite index with R as the set of right coset representatives. Then H is generated by the set $B = \{r_1 a r_2^{-1} \in H \mid a \in A, r_1, r_2 \in R\}$.

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Since $\langle B \rangle \leq H$ it follows that $\langle B \rangle = H$.

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for $i = 1$ **to** n **do**

while there are generators x, y fixing $1, 2, \ldots, i-1$
such that $i^x = i^y$ **do**

replace the pair x, y with the pair x, yx^{-1}.

end-while

end-for

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Theorem (Schreier-Sims)

Let $G < S_n$ be input by some generating set. In polynomial time we can compute a strong generating set $\cup T_i$ with the following properties:

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• In Schreier's lemma: computing orbit of i takes $n^2|S| \leq n^4$ operations, which also gives transversal R for $G_{[i]}$ in $G_{[i-1]}$.

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• In Schreier's lemma: computing orbit of i takes $n^2|S| \leq n^4$ operations, which also gives transversal R for $G_{[i]}$ in $G_{[i-1]}$.

• $|R| \cdot |S| = n^3$ operations for computing generating set of $G_{[i]}$. Applying reduce operation costs $|S|n \leq n^4$ operations.

Running Time Analysis

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Overall costs is $n^4 \cdot n$ plus $O(|S|n)$ operations. Each operation costs $O(n)$. Thus, $O(n^6 + |S|n^2)$ is the time.

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Lemma (Orbit Counting Lemma)

Let $G \leq S_n$ and $\mathrm{orb}(G)$ denote the number of orbits of G. Then

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Proof Define 0-1 matrix: $M_{g,i} = 1$ iff $i^g = i$. Equate row-wise and column-wise sums using $|O(i)| = |G|/|G_i|$.

Fixpoint free elements in G

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Problem: Given $G = \langle g_1, g_2, \ldots, g_k \rangle$ transitive, can we find a fixpoint free element in polynomial time?

Cameron-Cohen's Theorem

Theorem (CC92)

If $G \leq S_n$ is transitive then the group G has at least $|G|/n$ many fixpoint elements.

Proof

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|G| &\geq |G|/n + |G_1| + |G \setminus (A \cup G_1)|
\end{aligned}
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Which yields

$$|A| \geq |G|/n.$$

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Analysis Probability that we do not find a fixpoint free element in, say, n^2 trials is bounded by $(1 - 1/n)^{n^2} \approx e^{-n}$.

Deterministic Algorithm

- Let $\mathrm{move}(g) = n - \mathrm{fix}(g)$. Orbit counting lemma restated:

$$\mathbb{E}_{g \in G}[\mathrm{move}(g)] = \frac{1}{|G|} \sum_{g \in G} \mathrm{move}(g) = n - \mathrm{orb}(G).$$

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$$\mathbb{E}_{g \in G}[\mathrm{move}(g)] = \mathbb{E}_{\pi \in R} \mathbb{E}_{g \in G_1 \pi}[\mathrm{move}(g)].$$

- Thus, for some coset $G_1 \pi$ of G_1 in G we must have $\mathbb{E}_{g \in G_1 \pi}[\mathrm{move}(g)] > n - 1$.

Deterministic Algorithm Contd.

• For each coset representative $\pi \in R$ we explain how to efficiently compute $\mathbb{E}_{g \in G_1\pi}[\mathrm{move}(g)]$. Revisit the proof of the orbit counting lemma:
Recall $M_{g\pi,i} = 1$ iff $i^{g\pi} = i$.

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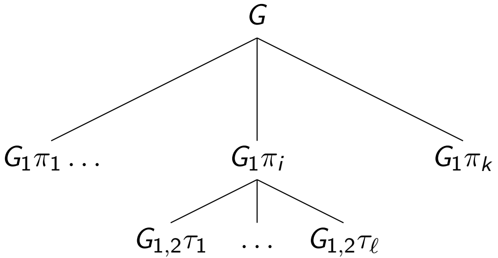
- Number of 1's in the i^{th} column is $|\{g \in G_1 \mid i^{g\pi} = i\}|$.

- This number is zero if i and $i^{\pi^{-1}}$ are in different G_1-orbits, and is $|G_{1,i}|$ otherwise.

Thus, using the Schreir-Sims algorithm we can compute all column sums efficiently, and hence also the above expectation in polynomial time.

Method of Conditional Probabilities



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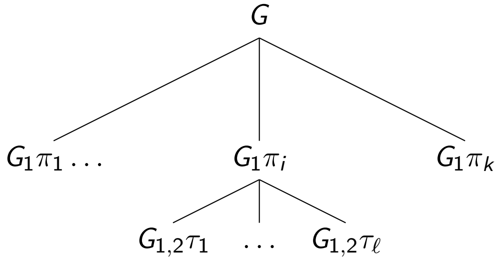

• At the d^{th} level of the tree compute the coset $G_{1,\ldots,d}\sigma$ that maximizes $\mathbb{E}_{g \in G_{1,\ldots,d}\sigma}[\mathrm{move}(g)]$.

Remarks and related questions

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• Thus, any irredudant base is of size $d \leq b \log n$.

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- Having picked $\{\alpha_1, \alpha_2, \ldots, \alpha_i\}$, pick α_{i+1} from an orbit of largest size in the pointwise stabilizer of $\{\alpha_1, \alpha_2, \ldots, \alpha_i\}$.

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If $H \leq G$ then H has an orbit of size at least $|H|^{1/b}$. Fixing a point in it makes $|H_\alpha| \leq |H|^{1/b}$.

• Thus, by picking $b \log \log n$ points $|G$ shrinks as $|G|^{(1-1/b)^{b \log \log n}} \approx e^b$. Pick $O(b)$ more points irredundantly.

Other Finite Groups

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• We cannot embed $\mathbb{F}_q^\times$ in S_n for small n if $q - 1$ has "large" prime factors.

References/further reading

- Peter J Cameron “Permutation Groups”, LMS Student Texts 45, Cambridge Univ Press.

- Eugene M Luks “Permutation groups and polynomial-time computation”, in Groups and Computation, DIMACS series in Discrete Mathematics and Theoretical Computer Science 11 (1993), 139-175. Available online.

- Laszlo Babai “Local expansion of vertex-transitive graphs and random generation in finite groups”.

Other Finite Groups Contd.

Suppose $G = \langle g_1, g_2, \ldots, g_k \rangle$ where we assume *no structure* about G. What can we compute efficiently? Randomness helps.

- *Random subproducts* of $g_1, g_2, \ldots, g_k$ are elements of the form

$$g_1^{\epsilon_1} g_2^{\epsilon_2} \ldots g_k^{\epsilon_k}, \epsilon_i \in_R \{0, 1\}.$$

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Input $G = \langle g_1, g_2, \ldots, g_k \rangle$.

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A randomized test
Let $x = g_1^{\epsilon_1} g_2^{\epsilon_2} \ldots g_k^{\epsilon_k}$ and $y = g_1^{\mu_1} g_2^{\mu_2} \ldots g_k^{\mu_k}$ be two independent subproducts.
Accept iff $xy = yx$.

Testing Commutativity Contd.

- **Claim** If $H < G$ is a proper subgroup then

$$\mathrm{Prob}[g_1^{\epsilon_1} g_2^{\epsilon_2} \ldots g_k^{\epsilon_k} \notin H] \geq 1/2.$$

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$$\begin{aligned}\mathrm{Prob}[xy = yx] &\leq \mathrm{Prob}[x \in Z(G)] + \mathrm{Prob}[y \in C(x) \wedge x \notin Z(G)] \\ &\leq p + (1-p)/2 \\ &= (1+p)/2 \\ &\leq 3/4\end{aligned}$$

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Curious Fact If G is nonabelian then

$$\mathrm{Prob}_{x,y \in G}[xy = yx] \leq 5/8.$$

Erdös-Rényi Sequences

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A random subproduct $g_1^{e_1} g_2^{e_2} \ldots g_k^{e_k}$ is ε-uniform in G if for all $x \in G$

$$(1 - \varepsilon)/|G| \leq \mathrm{Prob}[g_1^{e_1} g_2^{e_2} \ldots g_k^{e_k} = x \leq (1 + \varepsilon)/|G|.$$

If $k \geq 2 \log |G| + 2 \log(1/\varepsilon) + \log(1/\delta)$ and $g_1, g_2, \ldots, g_k$ are randomly picked, then $g_1^{e_1} g_2^{e_2} \ldots g_k^{e_k}$ is ε-uniform in G with probability $1 - \delta$.

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Include an element $g_{k+1} \in C^{-1}Cg_j \setminus C^{-1}C$ to extend the sequence. As $Cg_{k+1} \cap C = \emptyset$ we have doubled the size of the cube.

Short straight-line programs

In $\ell \leq \log |G|$ steps we obtain elements $g_{k+1}, g_{k+2}, \ldots, g_{k+\ell} \in G$ such that for the cube

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we have

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Each $g \in G$ has a straight-line program of length

$$\sum_{i=0}^{\log|G|} (2(k+i)+1) + 2(k+\log|G|) = O((k+\log|G|)\log|G|)$$

in terms of the original generators.

THANKS!