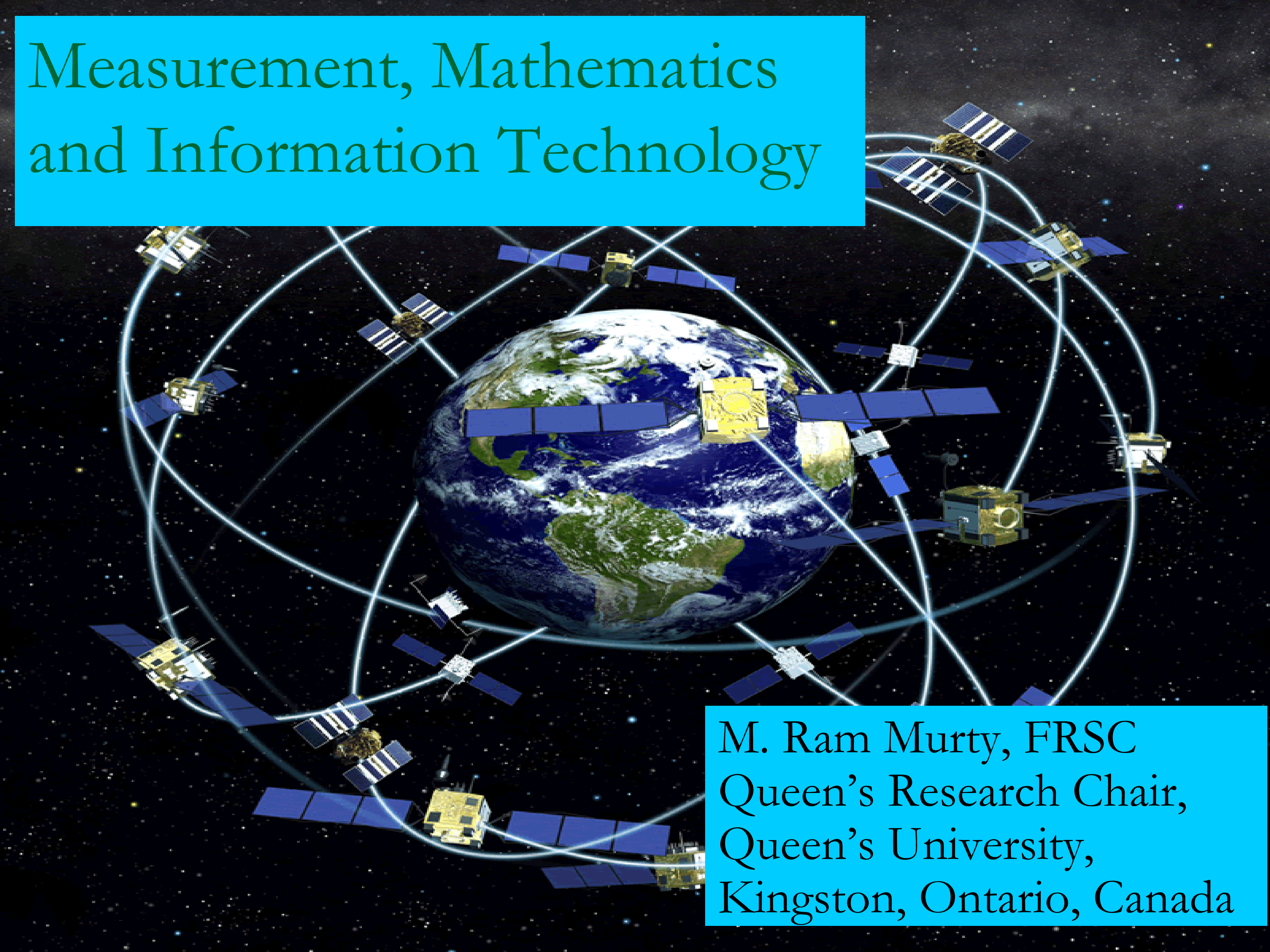
Measurement, Mathematics and Information Technology

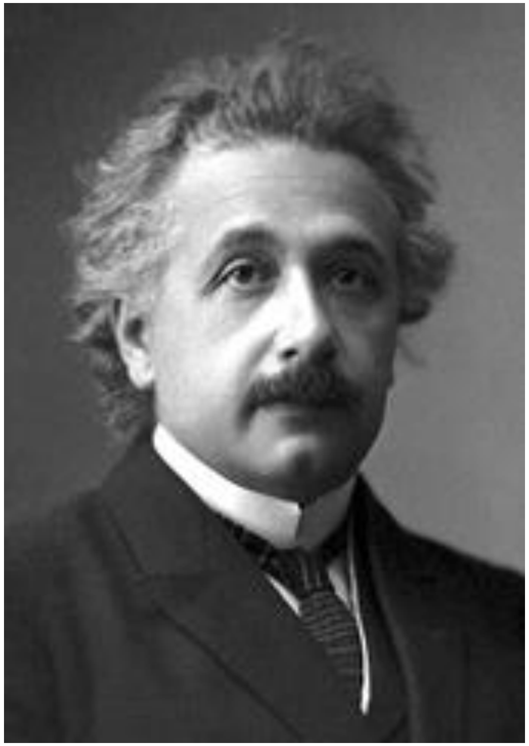


M. Ram Murty, FRSC
Queen’s Research Chair,
Queen’s University,
Kingston, Ontario, Canada

Can we measure everything?

- “Not everything that can be counted counts, and not everything that counts can be counted.”

However, some understanding emerges through measurement.



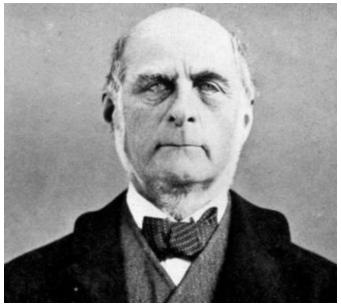
Albert Einstein (1879-1955)

Here is a quote from an article in the New York Times:

- “With the measurement system all but finalized, why are controversies over measurement still surfacing? Why are we still stymied when trying to measure intelligence, schools, welfare and happiness?” -NY Times, October 2011.
- The article goes on to say that there are two ways of measurement, one is “ontic” and the other “ontological.”
- The ontic way is what we are all familiar with, measuring things according to a scale. It is mathematical.
- The ontological way is more philosophical and we understand through inquiry, reflection and meditation.

The dangers of confusing the two methods

Let us look at the case of Francis Galton.



F. Galton (1822-1911)

- Here is what Wikipedia says about him.
- He was an English Victorian progressive, polymath, psychologist, anthropologist, eugenicist, tropical explorer, geographer, inventor, meteorologist, proto-geneticist, psychometrician, and statistician. He was knighted in 1909.

The creation of eugenics

- Inspired by Darwin’s 1859 theory of evolution, Galton proposed a theory of how to create a master race by measuring intelligence of races.
- In 1869, his book “Hereditary Genius” posited that human intelligence was inherited directly and diluted by “poor” breeding.
- “The natural ability of which this book treats is such as a modern European possesses in a much greater average share than men of the lower races.”
- There is a straight line between Galton’s method of measuring intelligence to Hitler’s views of a master race. Galton’s views also led to the horrible idea of IQ.
- Thus, we must know what can be measured and what cannot.

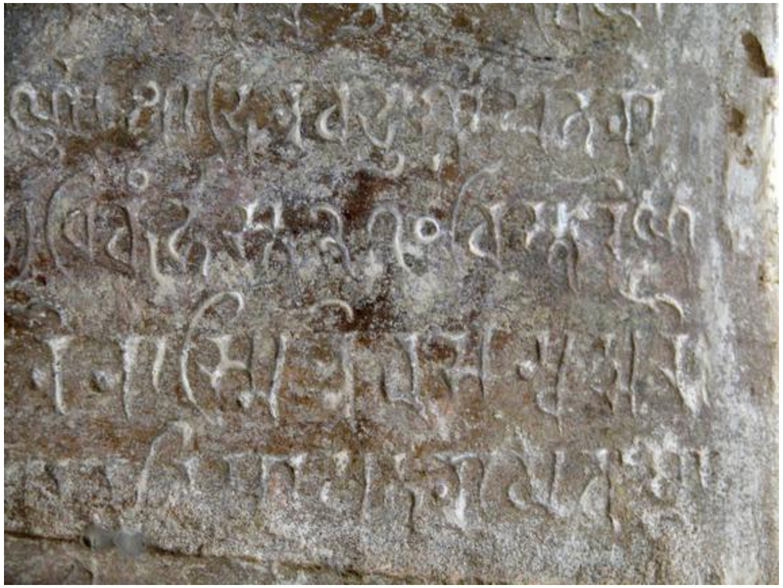
Everyday uses of measurement

- By measuring time, we are able to co-ordinate our daily activities.
- By measuring temperature, we can dress appropriately.
- By measuring cost, we can shop for the best deal.
- By measuring wind speeds and atmospheric currents, we can prepare for natural disasters.
- By measuring distance, we can plan our travel accordingly.
- All of these are "ontic" uses of measurement and all of them are invaluable in our daily life. These measurements have and continue to have a profound effect on civilizations.
- But they all need some basic knowledge of numbers.

However, there are many things that can be measured the “ontic” way.

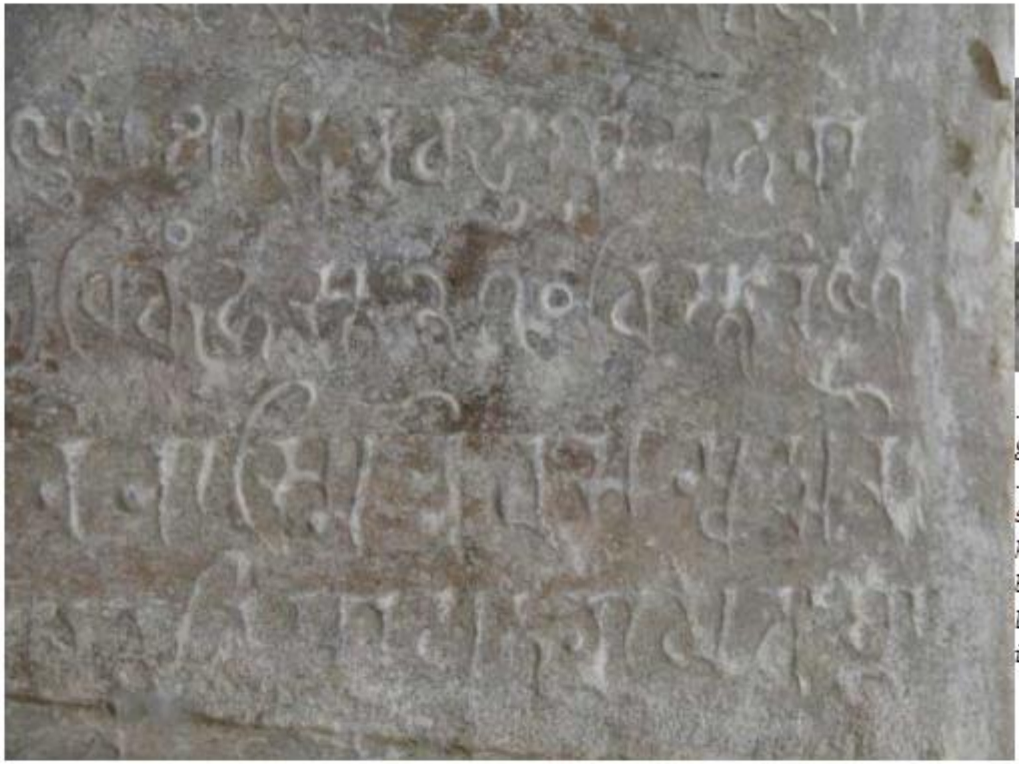
This is by using numbers.

- Many civilizations had a number system to count.
- Where does our decimal number system come from?
- India.
- More precisely, the decimal system goes back more than 1500 years to central India.
- In 7th century India, Brahmagupta wrote the first book that describes the rules of arithmetic using zero.



A portion of a dedication tablet in a rock-cut Vishnu temple in Gwalior built in 876 AD. The number 270 seen in the inscription features the oldest extant zero in India

The number 270



... the whole town gave to the temple ... which Alla, the son of Vaillabhatta, had caused to be built ... a piece of land ... 270 hastas in length ...

Gwalior

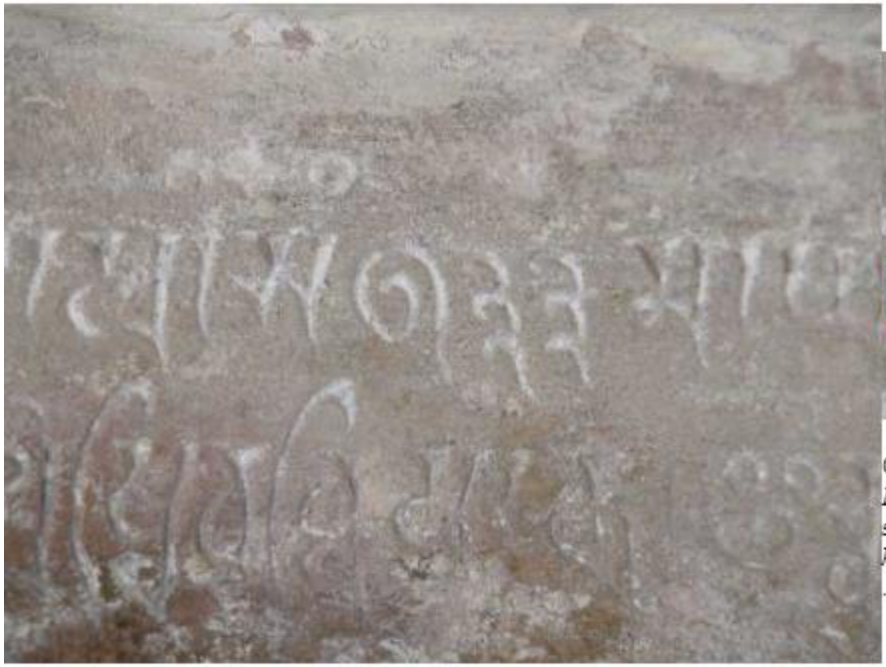
The origins of “zero” have been traced back to early Hinduism , Buddhism and Jainism where the concept of “nothingness” is equated with “nirvana” or the transcendental state.

- The rock inscription is part of the Vishnu temple is Gwalior.
- The Chinese and Babylonian civilizations had a place value number system. But it was the Indians that started to treat zero as a number.

The Vishnu Temple

- The defacement of the face probably occurred in the Mughal period (15th century).

Some more numbers on the temple walls

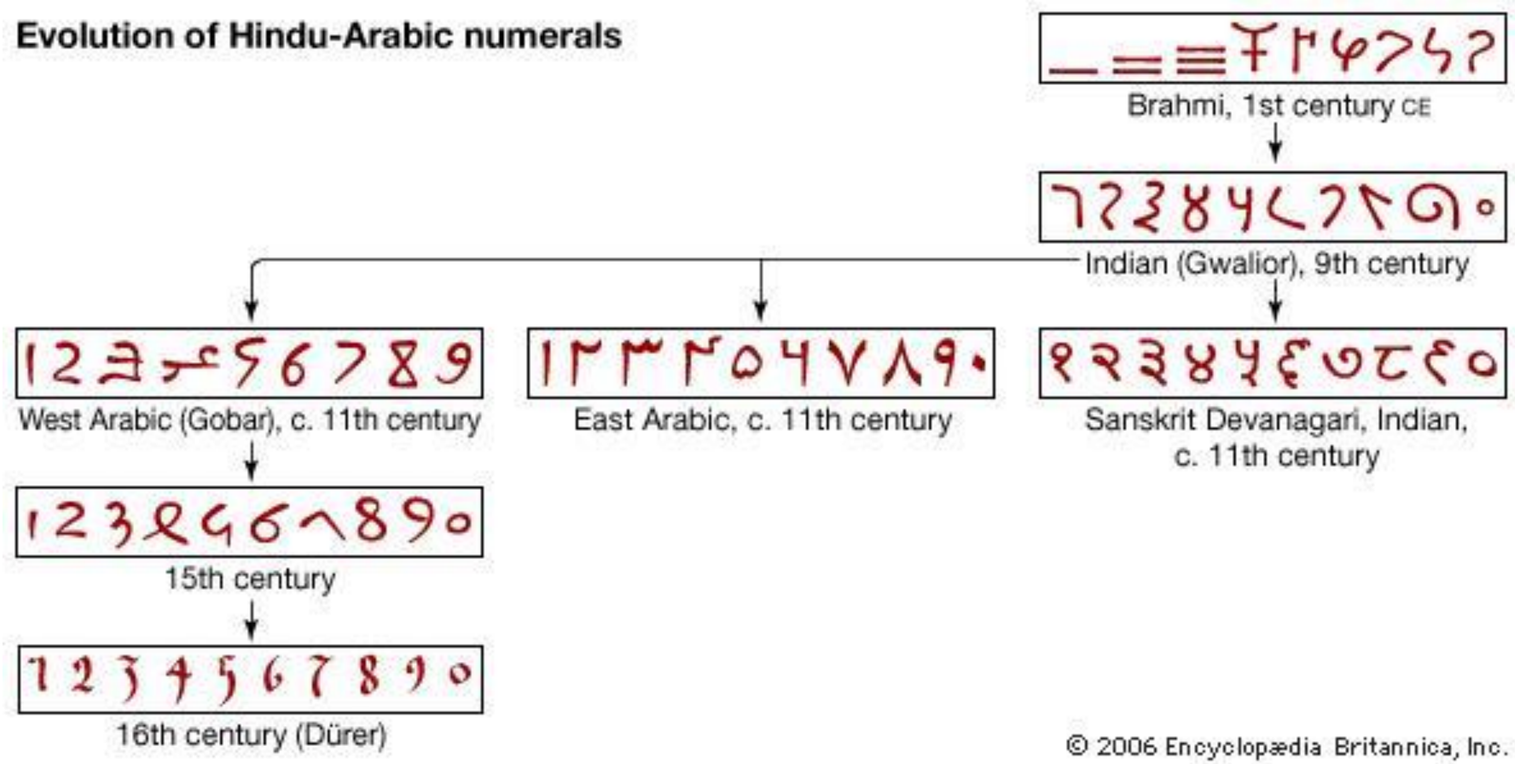


Om. Adoration to Vishnu! In the year 933, on the second day of the bright half of the month of Magha …

Some more …

... and 187 hastas in breadth, for a flower garden ...

Evolution of our number system

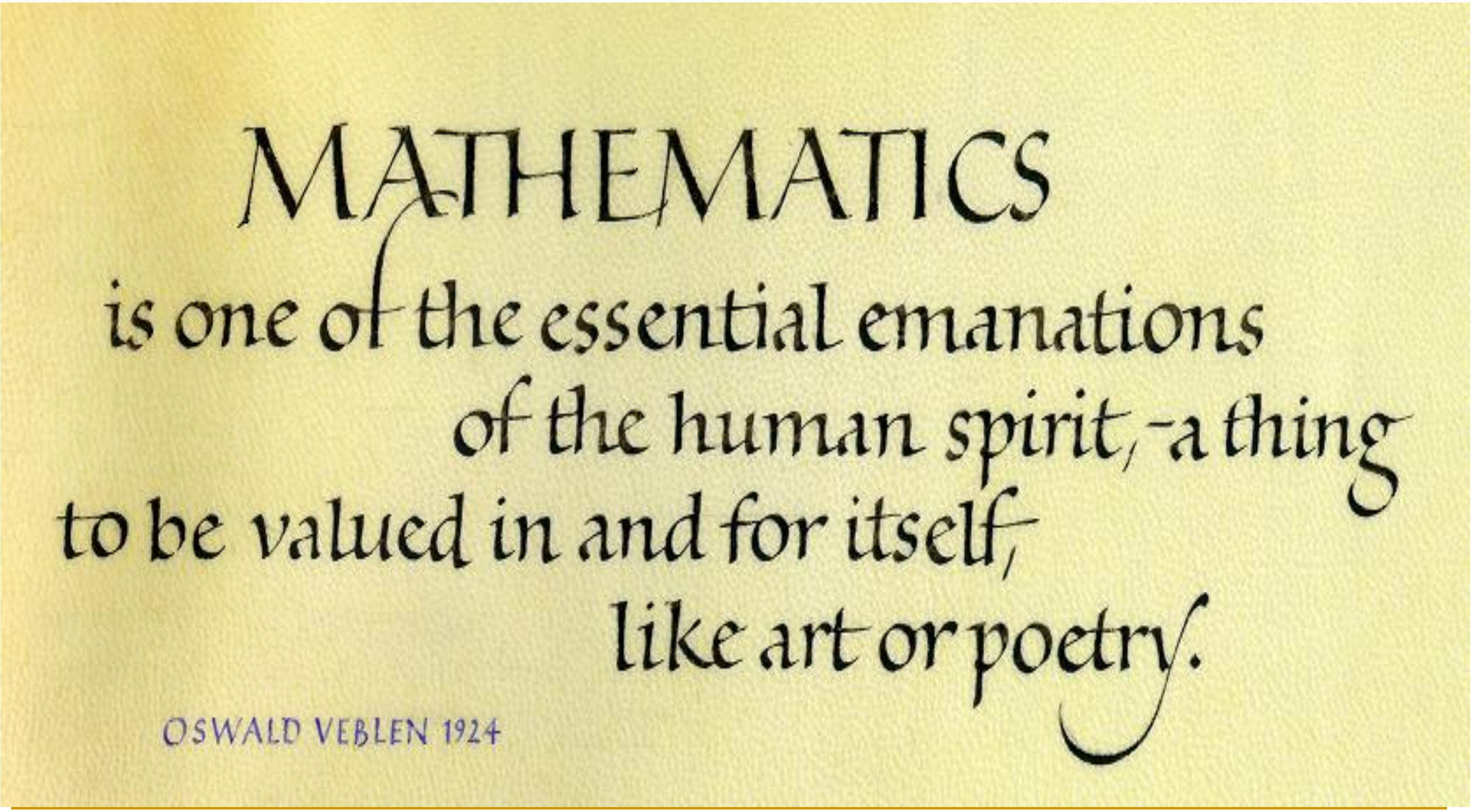


- Notice the similarity between the Gwalior system and our modern system of numerals.

The migration of the number system

- The familiar operations of numbers was developed by Brahmagupta around 600 CE.
- The number system then went to the middle east through Arab traders in the 8th century.
- Al-Khwarizmi wrote a book in 825 CE titled, “On the calculation with Hindu numerals”.
- The modern word “algorithm” comes from Al-Khwarizmi’s name.
- In 1202, Fibonacci took the number system from the Arabs and introduced in Europe but was not widely used until 1482, when printing came into vogue.
- This event animated the development of modern mathematics.

What is mathematics?

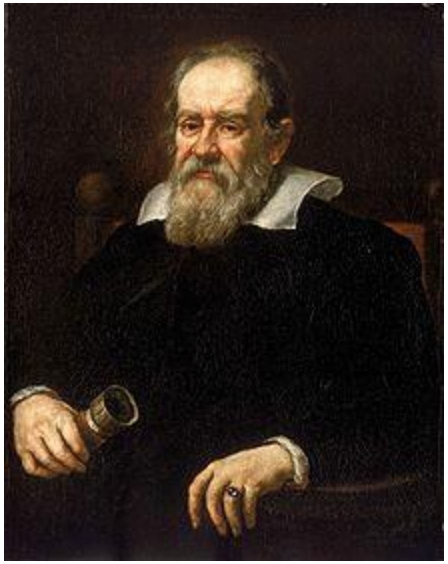


MATHEMATICS
is one of the essential emanations
of the human spirit, -a thing
to be valued in and for itself,
like art or poetry.

OSWALD VEBLEN 1924

Mathematics as the language of science

- "Nature's great book is written in the language of mathematics". - Galileo

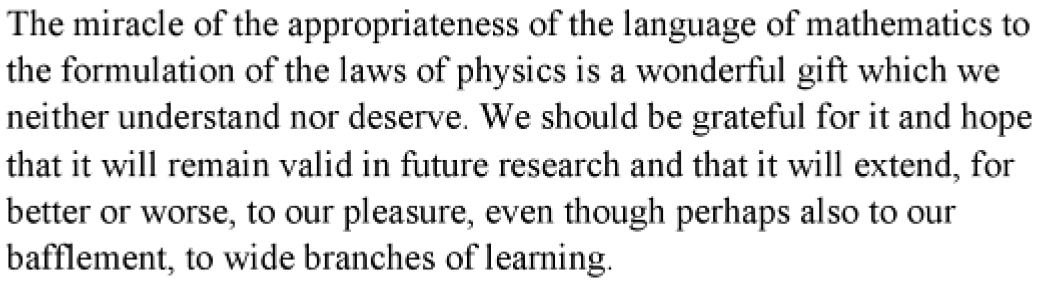


Galileo (1564-1642)

C.F. Gauss (1777-1855)

"Mathematics is the queen of science and number theory is the queen of mathematics." – C.F. Gauss

The unreasonable effectiveness of mathematics


The miracle of the appropriateness of the language of mathematics to the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.


Eugene Wigner
(1902-1995)

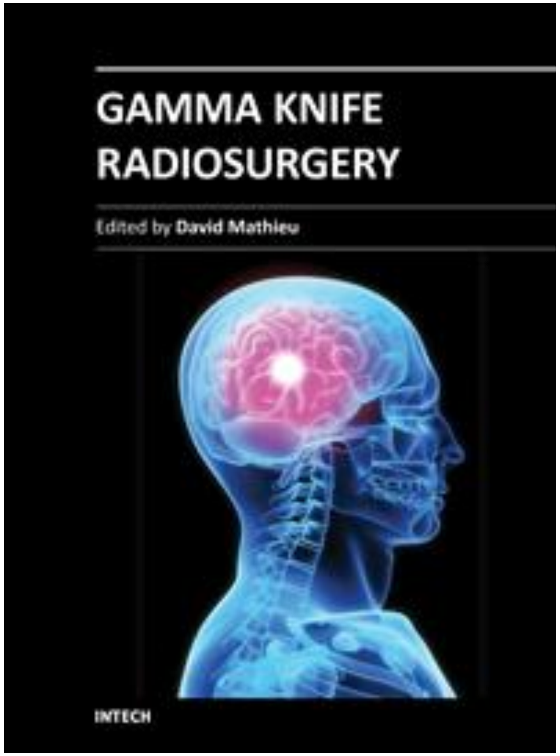
- Mathematics is now being applied to diverse fields of learning never imagined with remarkable success.
- In this talk, we will highlight some examples of this phenomenon.

Three examples of measurement

- We will discuss the mathematics behind:

position

importance



shape

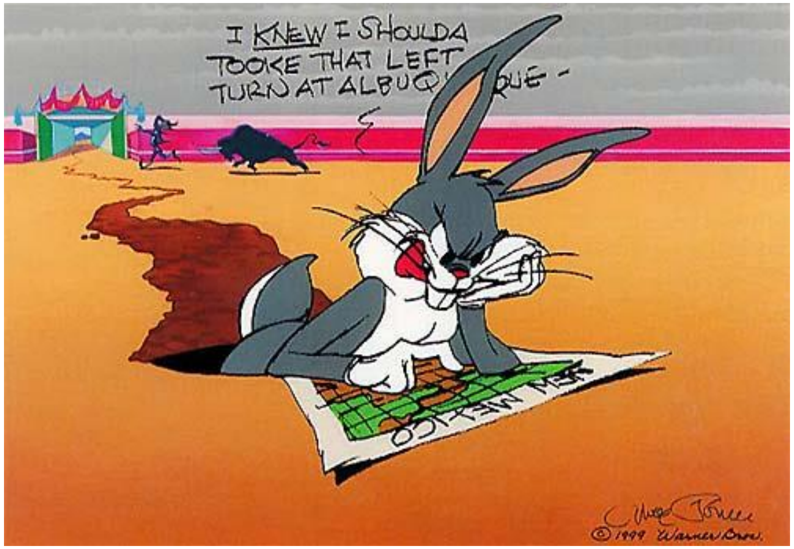
Who am I?

- This is the fundamental existential question and belongs in the realm of philosophy.
- GPS is concerned with the question "Where am I?" not as a philosophical question but as question in geography. What is my geographical position?

YESTERDAY, I DISCOVERED THAT I AM A CARTOON CHARACTER. I WAS UP ALL NIGHT TRYING TO DECIDE WHETHER I AM THE LINES OR THE SPACES IN BETWEEN.

"Who am I to question our government's policies? I'll tell you who I am! I'm Bernard A. Nesbitt, who reads the 'Times,' the 'Wall Street Journal,' the 'Post,' 'Newsweek,' 'Time,' 'Business Week,' 'U.S.News & World Report,' 'Look,' 'Life,' and the 'Saturday Evening Post.' That's who I am!"

The world without GPS



Mashable Comics
Written by @Matt_Silverman
Illustrated by Kiersten Essenpreis

The world with GPS

GPS: Satellites and Receivers

- Each satellite sends signals indicating its position and time.

Satellites and signals

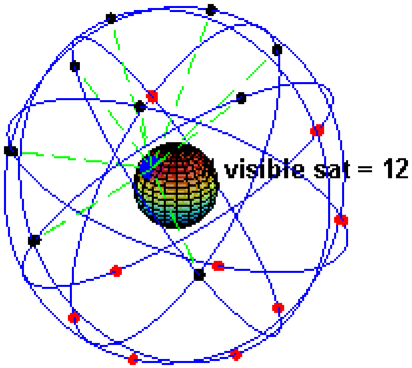
- Each satellite of the network sends a signal indicating its position and the time of the transmission of the signal.
- Since signals travel at the speed of light, the receiver can determine the radial distance of the satellite from the receiver based on the time it took to receive the signal since each receiver also has a clock.
- Many think that the receivers transmit information to the satellites, whereas in reality, it is the other way around.
- The receiver then uses basic math to determine its position.

Spheres

- If the receiver is R units away from satellite A, then the receiver lies on a sphere of radius R centered at A.
- A suitably positioned second satellite B can be used to determine another sphere, and the intersection of these two spheres determines a circle.
- A third satellite can be used to narrow the position to two points, and finally, a fourth, not coplanar with the other three, can be used to pinpoint the position of the receiver.

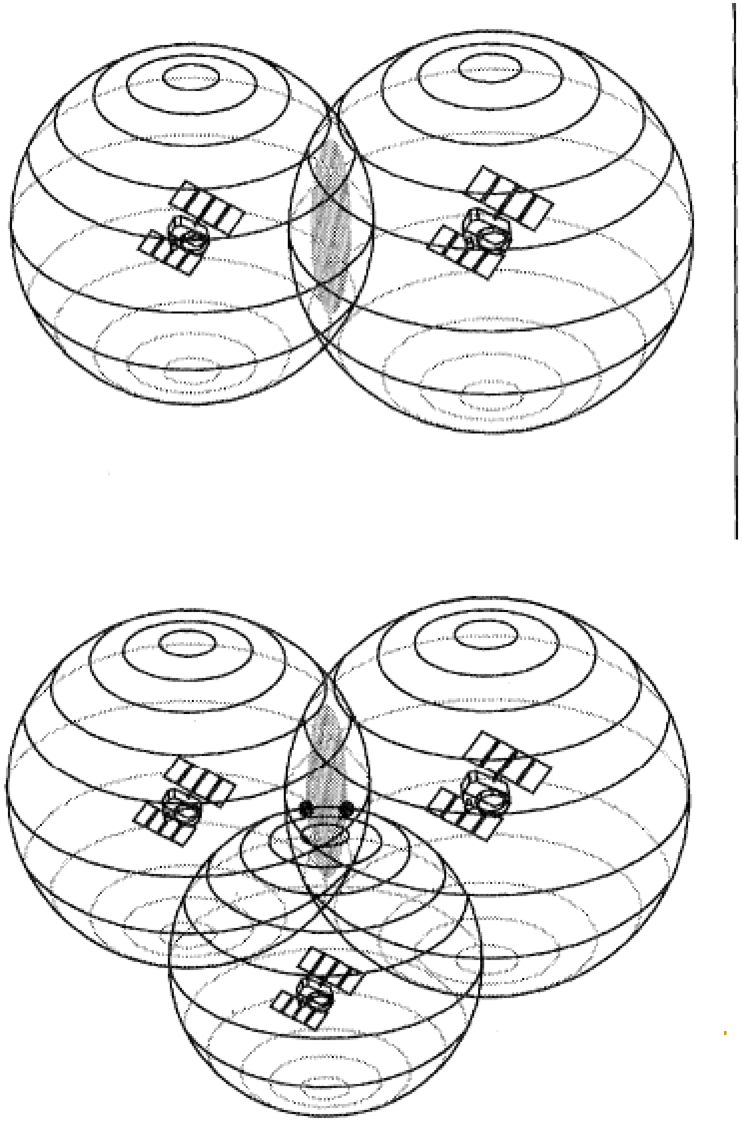
Satellites in orbit

- This is an animation of 24 GPS satellites with 4 satellites in each of 6 orbits. It shows how many satellites are visible at any given time. This ensures redundancy to ensure accuracy.



The mathematics of GPS

- The intersection of two spheres is either empty or a circle.
- The circle will intersect a third sphere in at most two points.
- This geometric fact is the basis of GPS since other factors can be used to eliminate one of the two points as being an irrelevant solution to the problem.



Equations for spheres

- Each satellite determines a radial distance to the receiver. Using Euclidean co-ordinates, let us denote the position of the receiver by (x,y,z) (which is unknown) and the position of the first satellite by (a_1, b_1, c_1) (which is known) and the radial distance by r_1. Then:

$$(x-a_1)^2+(y-b_1)^2+(z-c_1)^2=r_1^2$$

A second satellite

- A second satellite sends a signal to the receiver and determines another radial distance r_2. If the center of the second satellite is (a_2 , b_2 , c_2) then the unknown co-ordinates (x,y,z) lie on the sphere:

$$(x-a_2)^2+(y-b_2)^2+(z-c_2)^2=r_2^2$$

Similarly from a third satellite:

$$(x-a_3)^2+(y-b_3)^2+(z-c_3)^2=r_3^2$$

Solving three equations in three unknowns

$$(x-a_1)^2+(y-b_1)^2+(z-c_1)^2=r_1^2$$

$$(x-a_2)^2+(y-b_2)^2+(z-c_2)^2=r_2^2$$

$$(x-a_3)^2+(y-b_3)^2+(z-c_3)^2=r_3^2$$

- This is not a linear system. However, if we subtract the third from the first and the second from the first, we get two linear equations. Thus, our system is now of the form:

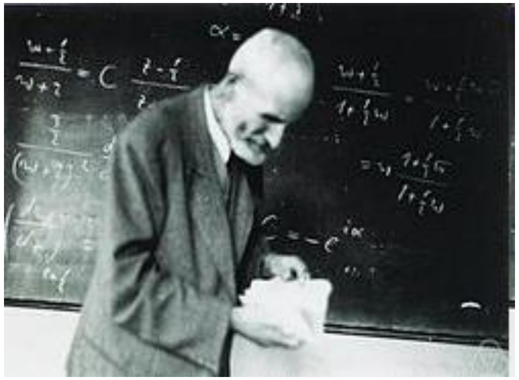
$$\begin{aligned}2(a_3-a_1)x+2(b_3-b_1)y+2(c_3-c_1)z&=A_1\\2(a_3-a_2)x+2(b_3-b_2)y+2(c_3-c_2)z&=A_2\\(x-a_3)^2+(y-b_3)^2+(z-c_3)^2&=r_3^2\end{aligned}$$

The first two equations determine x and y in terms of z via Cramer's rule in linear algebra. These are then plugged into the third giving us a quadratic equation in z. This gives two solutions for x,y,z but only one of these corresponds to a point on the surface of the earth, which determines the position of the receiver uniquely.

How Google works

- Google has become indispensable that many don't realize the non-trivial mathematics behinds its workings.
- The essential idea comes from a theorem of Frobenius and Perron dealing with Markov chains.

Georg Frobenius (1849-1917)



O. Perron (1880-1975)

A.A. Markov (1856-1922)

Google AS A NOUN!

"They're encyclopedias, Timmy. . . they're an early form of **Google**."

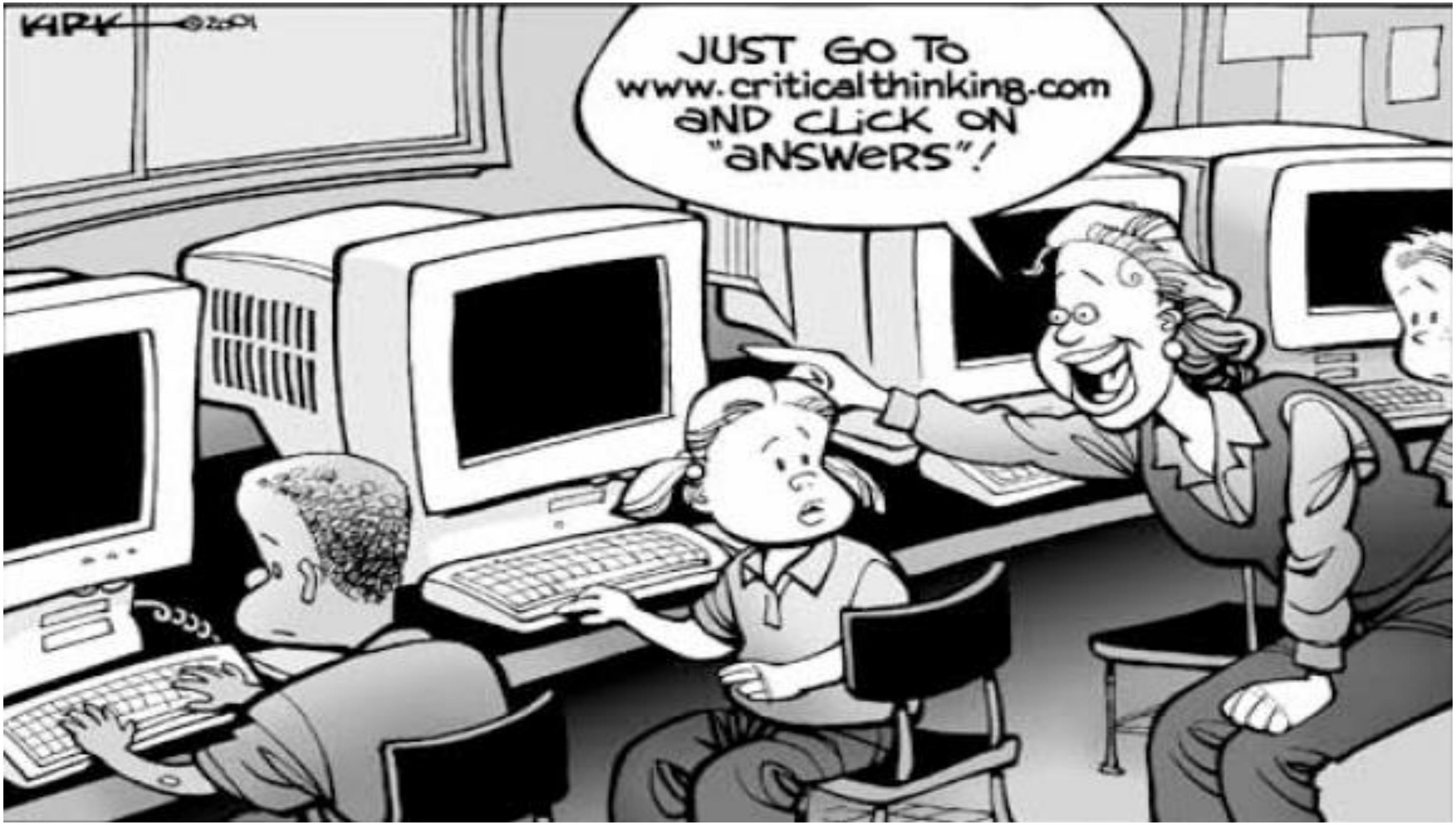
[redacted]

"I'M STUCK. CHECK IT OUT ON GOOGLE."

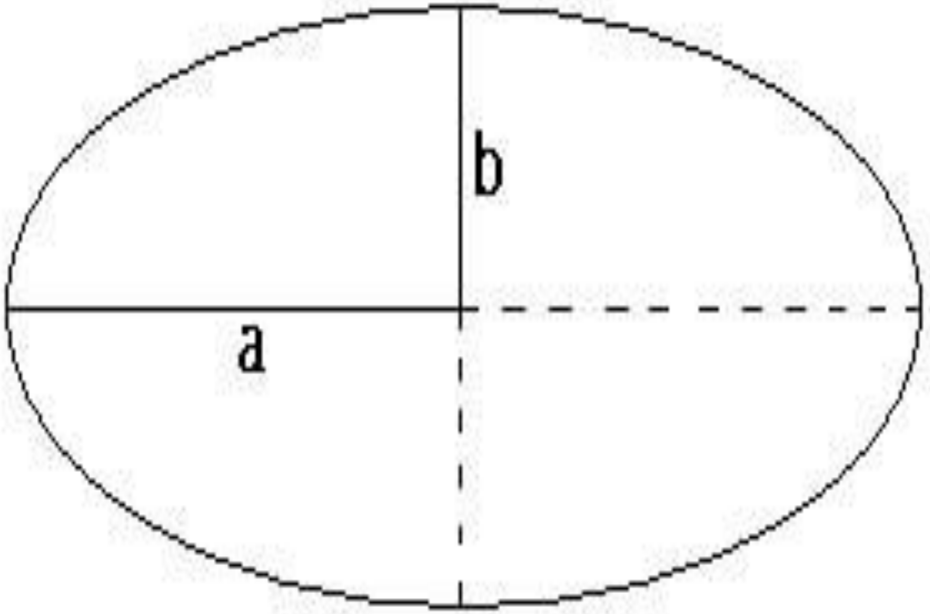
Google IS NOW A VERB!

"YOU'VE STUMPED ME WITH THAT QUESTION. I THINK THAT'S SOMETHING YOU NEED TO GOOGLE."

Google AS AN ORACLE!



From: gomath.com/geometry/ellipse.php



Area and Perimeter of Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\cancel{\text{Perimeter} = 2\pi\sqrt{\frac{a^2 + b^2}{2}}}$$

$$\text{Area} = \pi ab \checkmark$$

Metric mishap causes loss of Mars orbiter (Sept. 30, 1999)

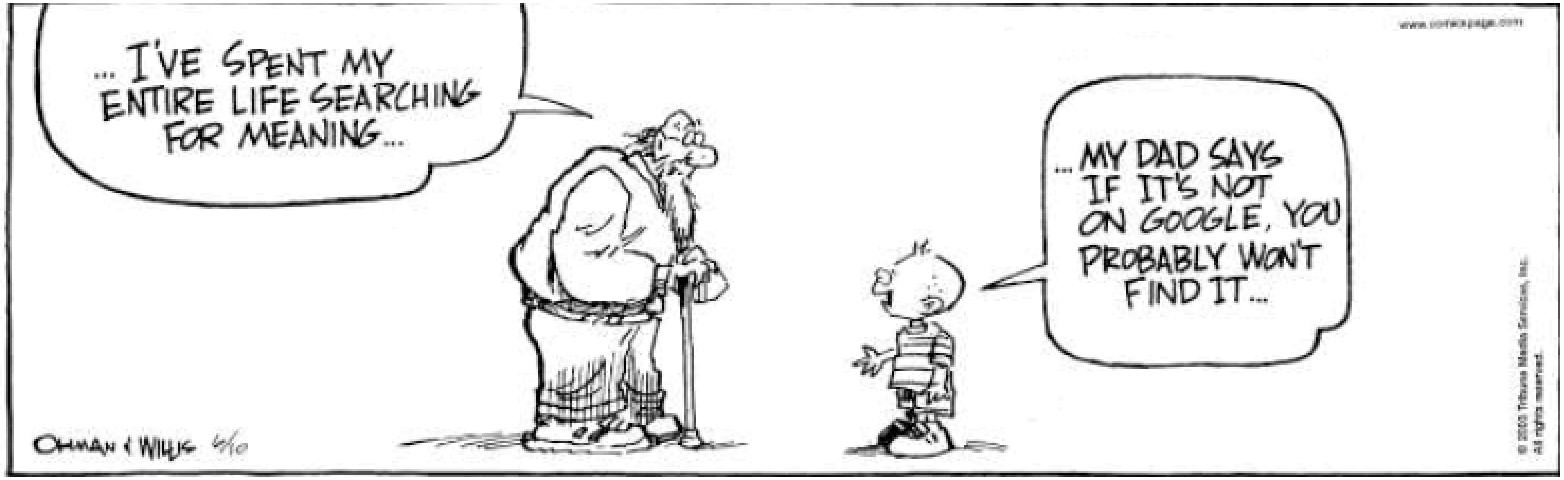
CNN NASA lost a 125 million Mars orbiter because a Lockheed Martin engineering team used English units of measurement while the agencys team used the more conventional metric system for a key spacecraft operation, according to a review finding released Thursday.

The units mismatch prevented navigation information from transferring between the Mars Climate Orbiter spacecraft team in at Lockheed Martin in Denver and the flight team at NASAs Jet Propulsion Laboratory in Pasadena, California.

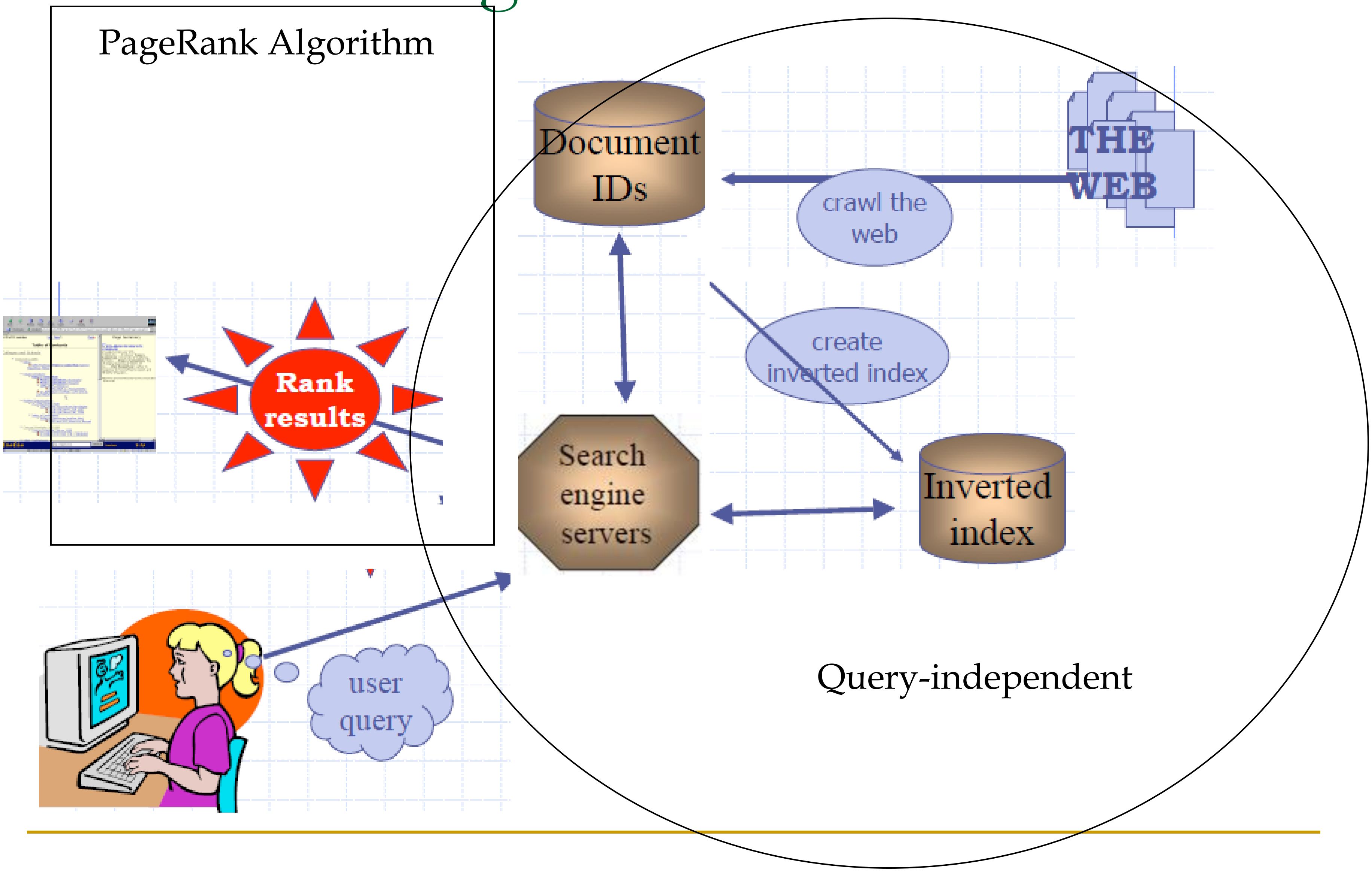
Lockheed Martin helped build, develop and operate the spacecraft for NASA. Its engineers provided navigation commands for Climate Orbiters thrusters in English units although NASA has been using the metric system predominantly since at least 1990.

(NASA)

The limitations of Google!



The web at a glance



PageRank Algorithm

Document IDs

crawl the web

THE WEB

create inverted index

Rank results

Search engine servers

Inverted index

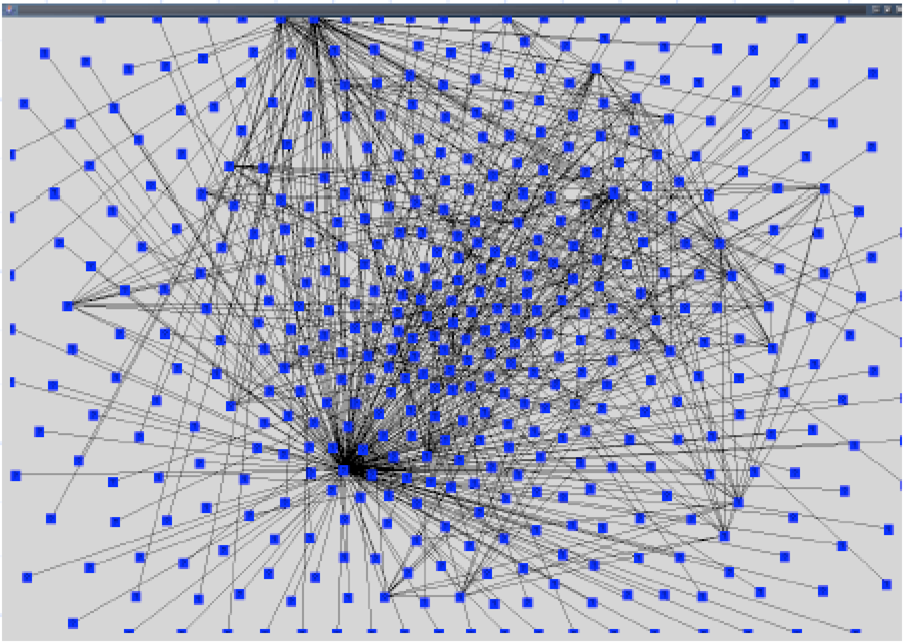
user query

Query-independent

The web is a directed graph

- The nodes or vertices are the web pages.
- The edges are the links coming into the page and going out of the page.

This graph has more than 10 billion vertices and it is growing every second!



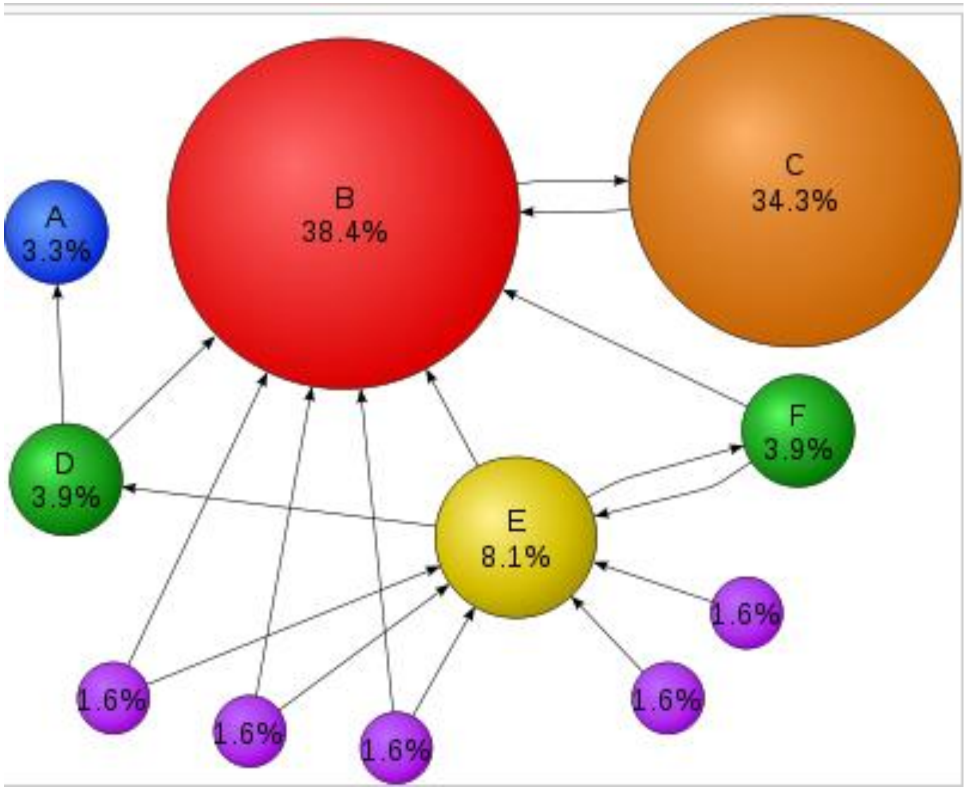
The PageRank Algorithm

- PageRank Axiom: A webpage is important if it is pointed to by other important pages.
- The algorithm was patented in 2001.

Sergey Brin and Larry Page

Example

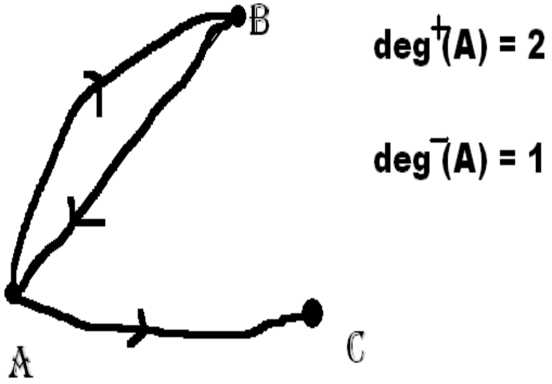
- C has a higher rank than E, even though there are fewer links to C since the one link to C comes from an "important" page.



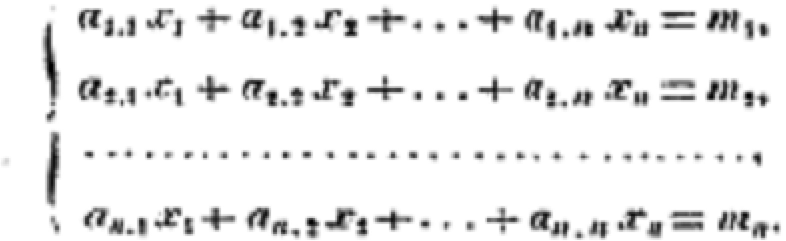
Cartoon illustrating basic principle of PageRank

Mathematical formulation

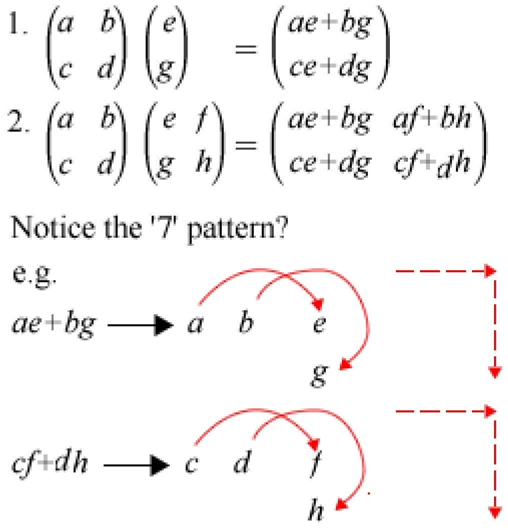
- Let r(J) be the “rank” of page J.
- Then r(K) satisfies the equation $r(K) = \Sigma_{J \to K}\, r(J)/\deg^{+}(J)$, where $\deg^{+}(J)$ is the outdegree of J.



Matrix multiplication



$$\left\{\begin{array}{l} a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n = m_1, \\ a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n = m_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ a_{n,1}x_1 + a_{n,2}x_2 + \dots + a_{n,n}x_n = m_n. \end{array}\right.$$



1. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e \\ g \end{pmatrix} = \begin{pmatrix} ae+bg \\ ce+dg \end{pmatrix}$

2. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$

Notice the '7' pattern?

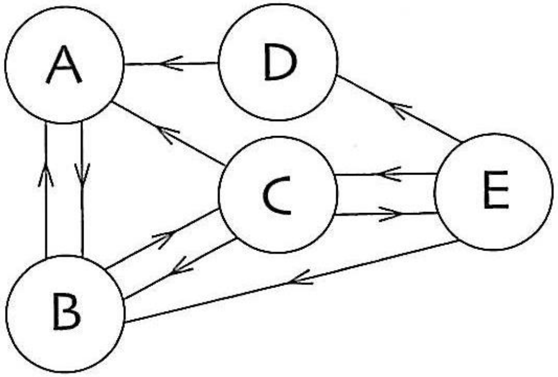
e.g.

H. Grassman
(1809-1877)

Factoid:
The word “matrix”
comes from the
Sanskrit word “matr”
which is the root word
for “mother”. It was coined by
Herman Grassman who was both a
Sanskrit scholar and a mathematician.

The web and Markov chains

- Let p_{uv} be the probability of reaching node u from node v.
- For example, p_{AB}=1/2 and p_{AC}=1/3 and p_{AE}=0.



$$P = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{ccccc} A & B & C & D & E \end{array}$$

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & 1 & 0 \\ 1 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 & 0 \end{pmatrix} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array}$$

Notice the columns add up to 1.
Thus, (1 1 1 1 1)P=(1 1 1 1 1).
P^t has eigenvalue 1

P is called the transition matrix.

Markov process

- If a web user is on page C, where will she be after one click? After 2 clicks? … After n clicks?

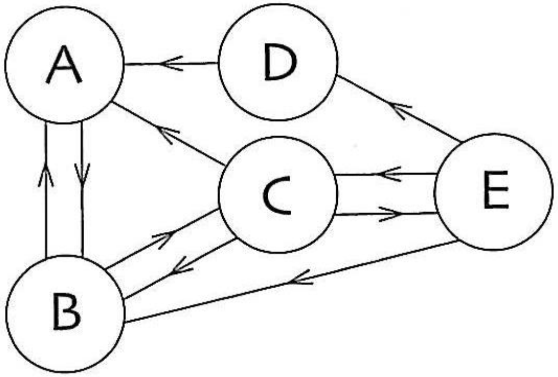
$$p^0 = \begin{pmatrix} p(X_0 = A) \\ p(X_0 = B) \\ p(X_0 = C) \\ p(X_0 = D) \\ p(X_0 = E) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$p^1 = \begin{pmatrix} p(X_1 = A) \\ p(X_1 = B) \\ p(X_1 = C) \\ p(X_1 = D) \\ p(X_1 = E) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & 1 & 0 \\ 1 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 0 \\ 0 \\ \frac{1}{3} \end{pmatrix},$$

$$p^2 = \begin{pmatrix} p(X_2 = A) \\ p(X_2 = B) \\ p(X_2 = C) \\ p(X_2 = D) \\ p(X_2 = E) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & 1 & 0 \\ 1 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 0 \\ 0 \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ \frac{4}{9} \\ \frac{5}{18} \\ \frac{1}{9} \\ 0 \end{pmatrix}.$$

After n steps, $P^n p^0$.

A.A. Markov (1856-1922)



Eigenvalues and eigenvectors

- A vector v is called an eigenvector of a matrix P if Pv = λv for some number λ.
- The number λ is called an eigenvalue.
- One can determine practically everything about P from the knowledge of its eigenvalues and eigenvectors.
- The study of such objects is called linear algebra and this subject is more than 100 years old.

Eigenvalues and eigenvectors of P

$$\Delta_{P^t}(\lambda) = \det(\lambda I - P^t) = \det(\lambda I - P)^t = \det(\lambda I - P) = \Delta_P(\lambda),$$

- Therefore, P and P^t have the same eigenvalues.
- In particular, P also has an eigenvalue equal to 1.

Theorem of Frobenius

- All the eigenvalues of the transition matrix P have absolute value ≤ 1.
- Moreover, there exists an eigenvector corresponding to the eigenvalue 1, having all non-negative entries.

Georg Frobenius (1849-1917)

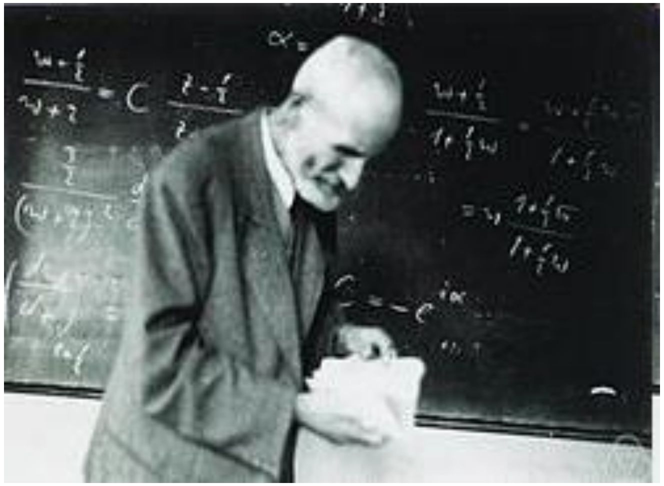
SIAM Review
Vol. 48, No. 3, pp. 569–581

The $25,000,000,000 Eigenvector:
The Linear Algebra behind Google[*]

Kurt Bryan[†]
Tanya Leise[‡]

Perron’s theorem

- Theorem (Perron): Let A be a square matrix with strictly positive entries. Let λ* = max{ |λ| : λ is an eigenvalue of A}. Then λ* is an eigenvalue of A of multiplicity 1 and there is an eigenvector with all its entries strictly positive. Moreover, |λ|< λ* for any other eigenvalue.



O. Perron (1880-1975)

Frobenius's refinement

- Call a matrix A irreducible if A^n has strictly positive entries for some n.
- Theorem (Frobenius): If A is an irreducible square matrix with non-negative entries, then λ^* is again an eigenvalue of A with multiplicity 1. Moreover, there is a corresponding eigenvector with all entries strictly positive.

Why are these theorems important?

- We assume the following concerning the matrix P:
- (a) P has exactly one eigenvalue with absolute value 1 (which is necessarily =1);
- (b) The corresponding eigenspace has dimension 1;
- (c) P is diagonalizable; that is, its eigenvectors form a basis.
- Under these hypothesis, there is a unique eigenvector v such that $Pv = v$, with non-negative entries and total sum equal to 1.
- Frobenius's theorem together with (a) implies all the other eigenvalues have absolute value strictly less than 1.

Computing $P^n p^0$.

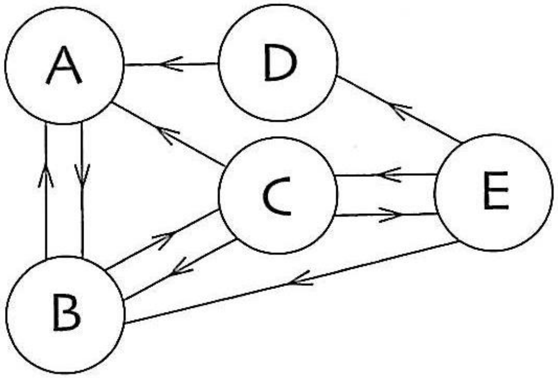
- Let $v_1, v_2, \ldots, v_5$ be a basis of eigenvectors of P, with v_1 corresponding to the eigenvalue 1.
- Write $p^0 = a_1 v_1 + a_2 v_2 + \ldots + a_5 v_5$.
- It is not hard to show that $a_1=1$.
- Indeed, $p^0= a_1 v_1 + a_2 v_2 + \ldots + a_5 v_5$
- Let $J=(1,1,1,1,1)$.
- Then $1 = J\,p^0= a_1 Jv_1 + a_2 Jv_2 + \ldots + a_5 Jv_5$
- Now $Jv_1=1$, by construction.
- For $i \geq 2$, $J(Pv_i) = (JP)v_i = Jv_i$. But $Pv_i = \lambda_i v_i$.
- Hence $\lambda_i Jv_i = Jv_i$. Since $\lambda_i \neq 1$, we get $Jv_i = 0$.
- Therefore $a_1=1$.

Computing $P^n p^0$ continued

- $P^n p^0 = P^n v_1 + a_2 P^n v_2 + \dots + a_5 P^n v_5$
- $\quad = v_1 + \lambda_2^n a_2 v_2 + \dots + \lambda_5^n a_5 v_5$.
- Since the eigenvalues $\lambda_2, \dots, \lambda_5$ have absolute value strictly less than 1, we see that $P^n p^0 \to v_1$ as n tends to infinity.
- Moral: It doesn't matter what p^0 is, the stationary vector for the Markov process is v_1.

Returning to our example …

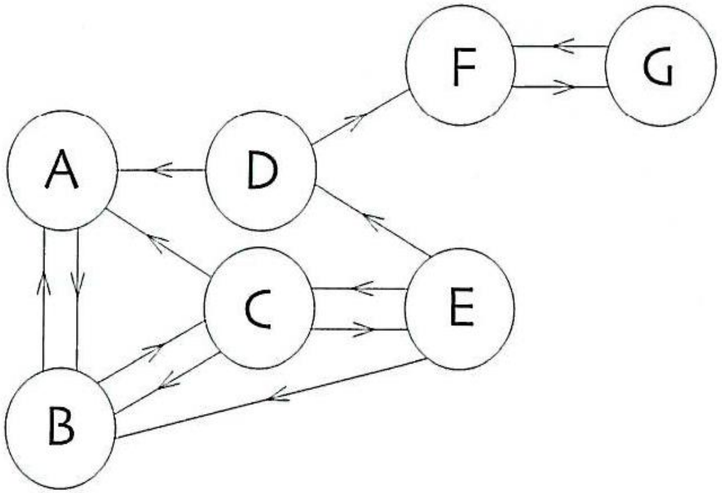
- The vector (12, 16, 9, 1, 3) is an eigenvector of P with eigenvalue 1.
- We can normalize it by dividing by 41 so that the sum of the components is 1.
- But this suffices to give the ranking of the nodes:B, A, C, E, D.



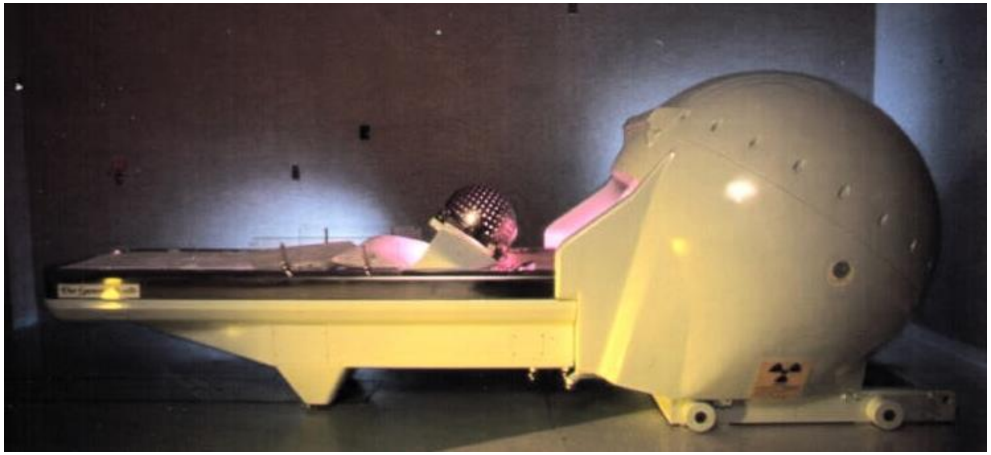
$$P = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccc} A & B & C & D & E & \\ 0 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & A \\ 1 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & B \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{3} & C \\ 0 & 0 & 0 & 0 & \frac{1}{3} & D \\ 0 & 0 & \frac{1}{3} & 0 & 0 & E \end{array}$$

Improved PageRank

- If a user visits F, then she is caught in a loop and it is not surprising that the stationary vector for the Markov process is (0,0,0,0,0, ½, ½)^t.
- To get around this difficulty, the authors of the PageRank algorithm suggest adding to P a stochastic matrix Q that represents the "taste" of the surfer so that the final transition matrix is P′ =xP + (1-x)Q for some 0≤x≤1.
- Note that P′ is again stochastic.
- One can take Q=J/N where N is the number of vertices and J is the matrix consisting of all 1′s.
- Brin and Page suggested x=.85 is optimal.

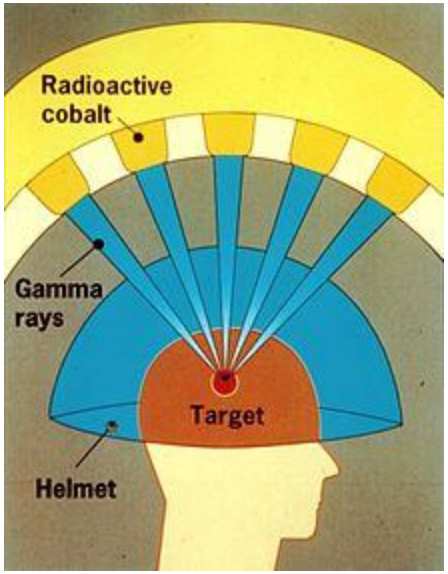


Radiosurgery and the mathematics of shapes



- Radiosurgery is also called gamma-knife surgery in the literature.

What is gamma-knife surgery?



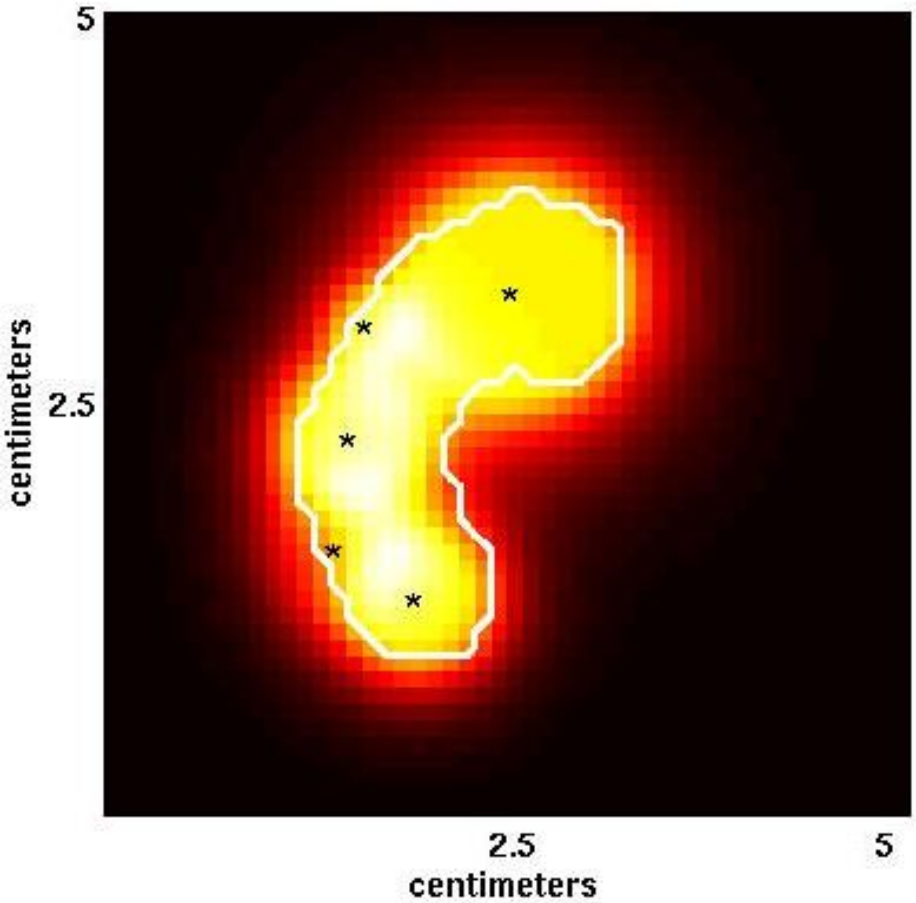
- It is a non-invasive medical procedure used to treat tumors, usually in the brain.
- This is called radiosurgery since it uses radiation to perform the surgery.
- 201 Cobalt gamma ray beams are arrayed in a hemisphere and aimed through a collimator to a common focal point.
- The patient’s head is positioned so that the tumor is the focal point.

The minimax problem

- Since the tumor maybe of irregular shape and spread over a region, the idea is to minimize the number of radiation treatments and maximize the portion of the area to be treated.
- When the beams are focused with the help of a helmet, they produce focal regions of various sizes.
- Each size of dose requires a different helmet and so the helmet needs to be changed when the dose radius needs to be changed.
- Since each helmet weighs 500 pounds, it is important to minimize the number of helmet changes.

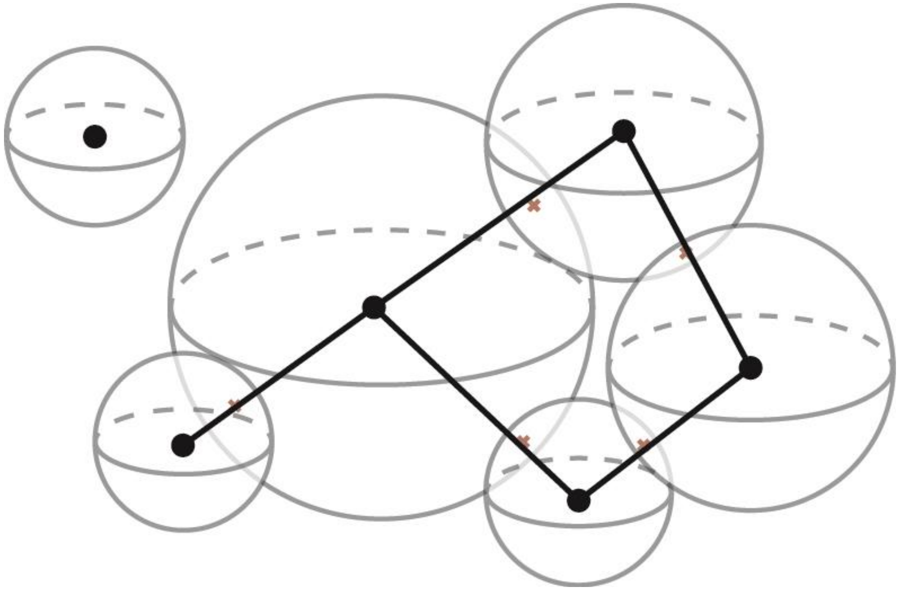
The mathematics of shapes

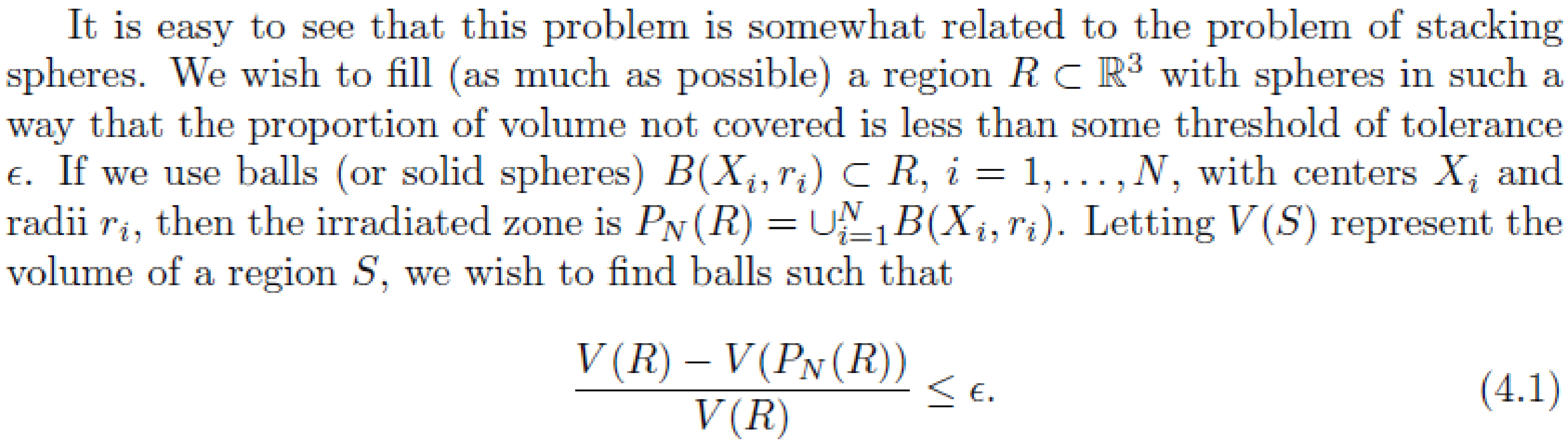
- Here is the target area on which the radiation is to be applied.
- Since the helmets have varying degrees of focal regions, several helmets have to be used.



Sphere packing problem

- Since we have spheres of different sizes and not all of the affected region can be targeted, the problem can be formulated mathematically as follows:





It is easy to see that this problem is somewhat related to the problem of stacking spheres. We wish to fill (as much as possible) a region $R \subset \mathbb{R}^3$ with spheres in such a way that the proportion of volume not covered is less than some threshold of tolerance ϵ. If we use balls (or solid spheres) $B(X_i, r_i) \subset R$, $i = 1, \ldots, N$, with centers X_i and radii r_i, then the irradiated zone is $P_N(R) = \cup_{i=1}^{N} B(X_i, r_i)$. Letting $V(S)$ represent the volume of a region S, we wish to find balls such that

$$\frac{V(R) - V(P_N(R))}{V(R)} \leq \epsilon. \qquad (4.1)$$

The skeleton of a region

- Let |X-Y| denote the Euclidean distance between two points in the plane or in space.

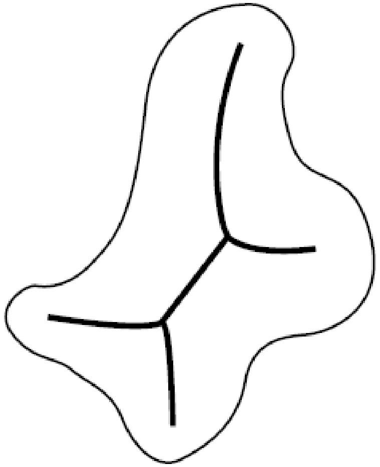


Fig. 4.1. The skeleton of a region.

Thus, if two points X and $Y \in \mathbb{R}^2$ have coordinates (x_1, y_1) and (x_2, y_2) respectively, then the distance between them is

$$|X - Y| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Definition 4.2 *Let R be a region of $\mathbb{R}^2$ (or $\mathbb{R}^3$) and let ∂R be its boundary. The* skeleton *of R, denoted by $\Sigma(R)$, is the following set of points:*

$$\Sigma(R) = \left\{ X^* \in R \middle| \begin{array}{c} \exists X_1, X_2 \in \partial R \text{ such that } X_1 \neq X_2 \text{ and} \\ |X^* - X_1| = |X^* - X_2| = \min_{Y \in \partial R} |X^* - Y| \end{array} \right\}.$$

Two dimensional skeletons

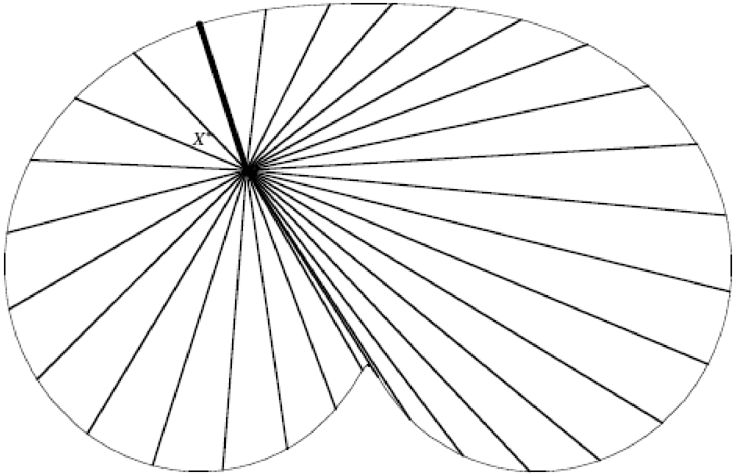
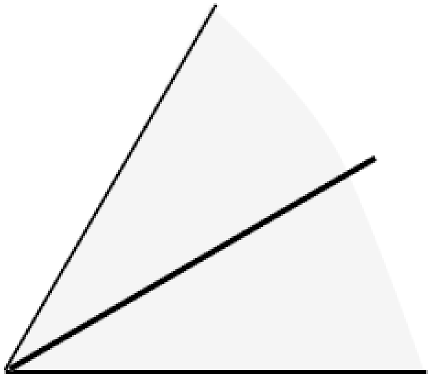


Fig. 4.2. Looking for the shortest distance between a point X^* and the boundary ∂R.

- We denote the skeleton of a region R by Σ(R).

Proposition 4.3 *Let* $X^* \in R$ *and* $d = \min_{Y \in \partial R} |X^* - Y|$. *Then* d *is the maximum radius such that* $B(X^*, d)$ *lies completely within* R, *i.e.,* $d = \max\{c > 0 : \ B(X^*, c) \subset R\}$. *The point* X^* *is in the skeleton* $\Sigma(R)$ *if and only if* $S(X^*, d) \cap \partial R$ *contains at least two points.*

Simple skeletons



(a) Skeleton of an angular region.

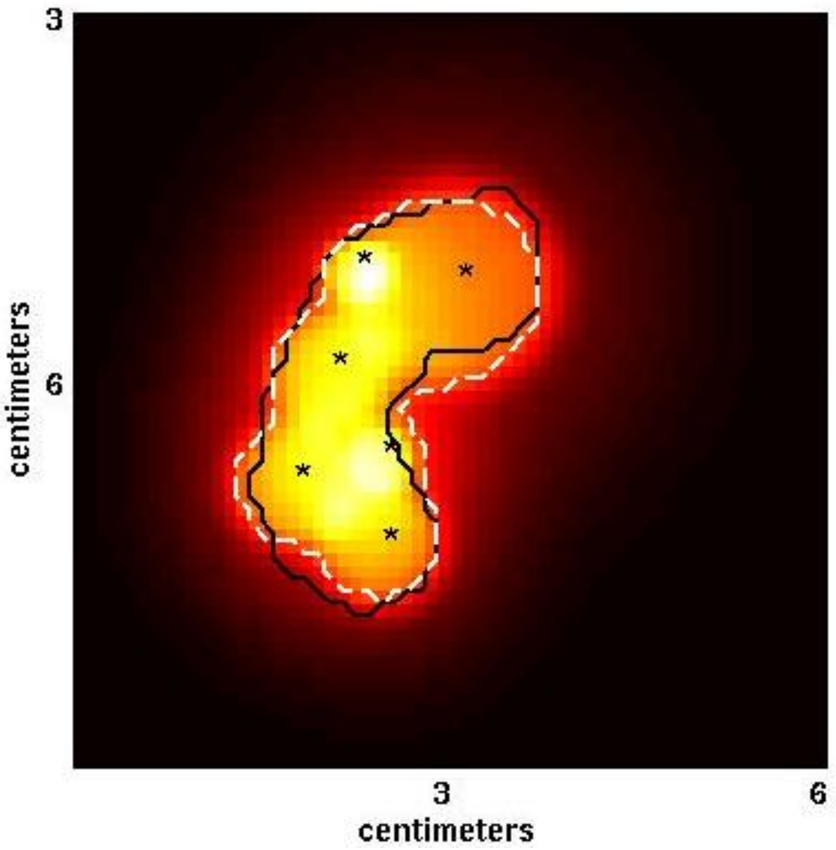
(b) Skeleton of a strip region.

- Given a region in R^2 we want to determine its skeleton since the centers of the focal regions will be situated along the skeleton.

(a) The skeleton

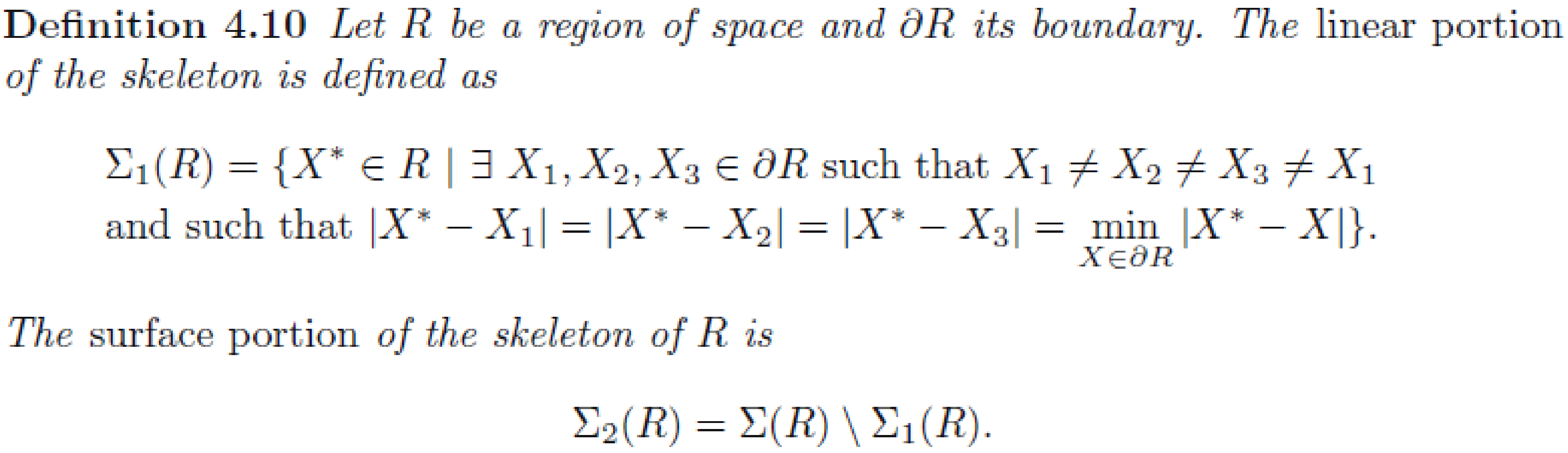
Skeletons in R^3

- The gamma rays will be focused on selected points along the skeleton of the region.



Three dimensional skeletons

- Our earlier definition of a skeleton applies in higher dimensions as well, and in particular to R^3. However, here we can distinguish two portions of the skeleton.



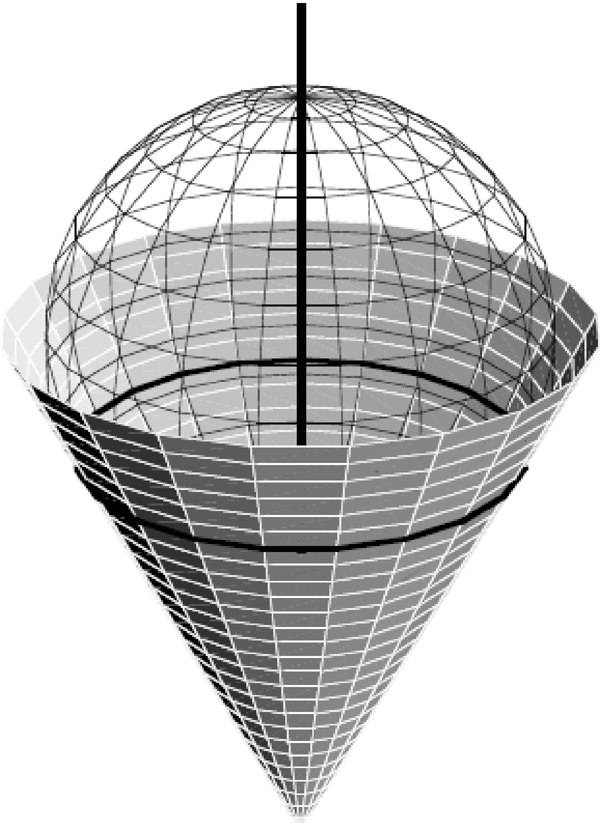
Definition 4.10 *Let R be a region of space and ∂R its boundary. The* linear portion *of the skeleton is defined as*

$$\Sigma_1(R) = \{X^* \in R \mid \exists\, X_1, X_2, X_3 \in \partial R \text{ such that } X_1 \neq X_2 \neq X_3 \neq X_1 \text{ and such that } |X^* - X_1| = |X^* - X_2| = |X^* - X_3| = \min_{X \in \partial R} |X^* - X|\}.$$

The surface portion *of the skeleton of R is*

$$\Sigma_2(R) = \Sigma(R) \setminus \Sigma_1(R).$$

Some simple examples

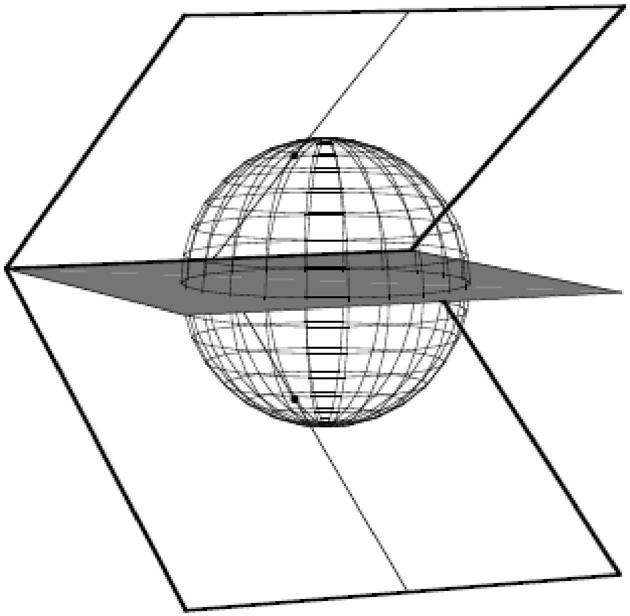


(a) The skeleton of a solid circular cone is given by its central axis

- While the region is the solid filled cone, only the boundary is shown as well as one maximal ball and its circle of tangency.

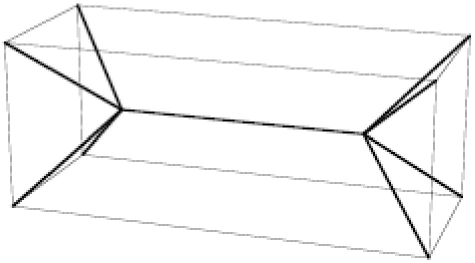
Skeleton of a wedge

- An infinite wedge consists of all points between two half-planes emanating from a common axis. A maximal sphere is shown with its points of tangency.

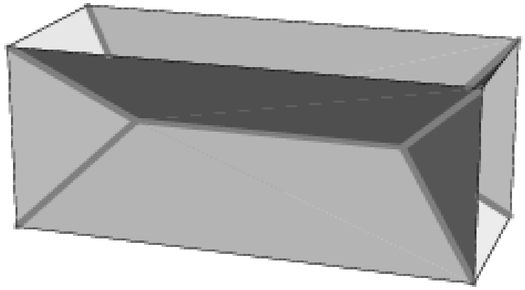


(b) The skeleton of an infinite wedge is given by the bisecting half-plane

Skeleton of a parallelepiped

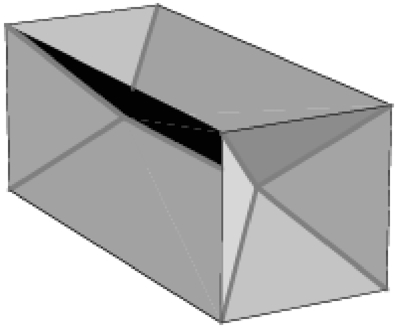


(a) The linear part of the skeleton



(b) The entire skeleton

- These examples are simple since the region is simple to describe. In general, the problem of finding the skeleton of a general region is based on computer algorithms.



(c) A second view of the skeleton

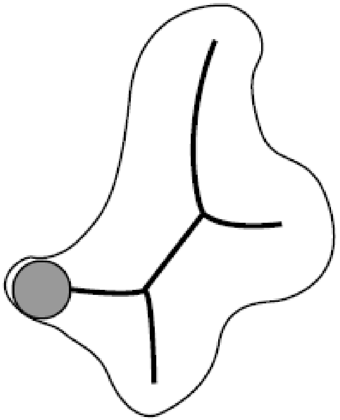
The optimal surgery algorithm

The underlying idea. Suppose that an optimal solution for a region R is given by

$$\cup_{i=1}^{N} B(X_i^*, r_i).$$

Then if $I \subset \{1, \ldots, N\}$, we must have that $\cup_{i \notin I} B(X_i^*, r_i)$ is an optimal solution for $R \setminus \cup_{i \in I} B(X_i^*, r_i)$

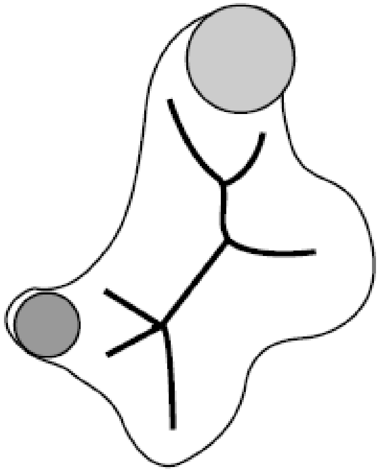
- Any dose in an optimal solution must be centered along the skeleton. If we have four sizes of doses, a<b<c<d (say), then the initial does should be at an extreme point of the skeleton.



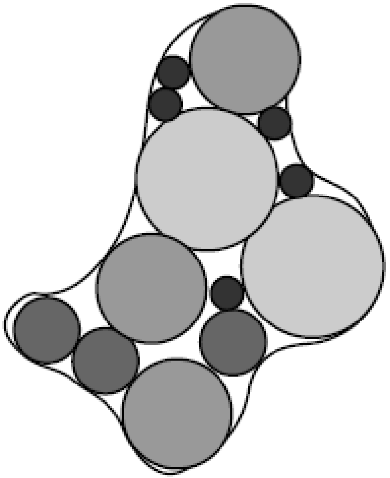
(a) A first dose of radius 4 mm

The iterative procedure

- After the first dose, the region has changed and we need to re-calculate the skeleton.



(b) The skeleton of the remaining region after two doses of radii 4 and 7 mm



(c) The entire region irradiated with doses of radii 2, 4, 7 and 9 mm

Summary

- GPS uses spherical geometry discovered 2000 years ago. It also uses relativity for accurate timing.

Where am I?

- Google uses the theory of Markov chains discovered 200 years ago.

What is "important"?

- Gamma knife radio surgery uses differential geometry discovered about 150 years ago.

What is a shape?

- The mathematics used is "pure" mathematics and when it was discovered, it was motivated by "aesthetic" considerations.

References

- Mathematics and Technology, by C. Rousseau and Y. Saint-Aubin, Springer, 2008.
- Google's PageRank and Beyond, The Science of Search Engines, A. Langville and C. Meyer, Princeton University Press, 2006.
- The 25 billion dollar eigenvector, K. Bryan and T. Liese, SIAM Review, 49 (2006), 569-581.

Mathematical genealogy

P.L.Chebychev
(1821-1894)

A.A.Markov
(1856-1922)

J.D.Tamarkin
(1888-1945)

D.H. Lehmer
(1905-1991)

H.M. Stark
(1939-

Thank you for your attention.

- Have a Goooooooooogle day!

© Randy Glasbergen
glasbergen.com

“I looked up your symptoms on Google.
If you want a second opinion, I can check Yahoo.”