PERMUTATIONS AND COMBINATIONS

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1. General Principles of Counting

Example 1. (a) Find the total number of three letter words. Here, by a *word*, we mean any string of three alphabets. (b) Now, find the number of such words which contain only vowels or only consonants.

Since there are 26 choices for each of the three letters in the word, the total number in part (a) is 26^3. In part (b), we subdivide this into two problems: counting those words which contain only vowels (5^3) and those which contain only consonants (21^3). The answer is then the sum $5^3 + 21^3$. □

Example 2. Find the number of 8 digit numbers formed only using the digits $4, 5, 6$ which are (a) even (b) divisible by 5 (c) divisible by 3.

(a) For such a number to be even, the units digit has to be even, hence 4 or 6. Thus there are two choices for the units digit, and 3 choices for each of the remaining 7 digits. Hence the total number of such numbers is $3^7 \times 2$. (b) Similar, with only one choice for units digit (namely 5). Thus, here the count is 3^7. (c) This is more complicated since all digits play a role in determining if a number is divisible by 3. We need the sum of all 8 digits to be a multiple of 3. We proceed as follows: first choose the first 7 digits arbitrarily (3^7 ways of doing this). The units digit is now uniquely determined by the requirement that the sum of digits is divisible by 3; this is because the set of allowed digits $\{4, 5, 6\}$ leave different remainders on division by 3, and exhaust all remainders. Thus, here again the count is 3^7. □

Problem 1. Redo part (c) of example 2 under the assumption that the allowed digits are now $1, 2, 3, 5, 7, 9$.

Problem 2. Pose your own variations of problem 1, and explore ways of solving them.

We used two general principles in the above examples:

(1) *The sum principle:* If the set of objects to be counted can be separated into two disjoint subsets, then the total number of objects is the sum of the numbers of objects in the two subsets.
(2) *The product principle:* If an object A can be chosen in m ways, and once A is chosen, an object B can be chosen in n ways, then the number of ways of choosing the ordered pair (A, B), i.e., first A, then B, is mn.

2. Permutations

Given a set S of objects, a *permutation* of S is a way of arranging these objects in a line. If S has n elements, a permutation can be formally defined as a function $f : \{1, 2, \cdots, n\} \to S$ such that f is injective (i.e., one-to-one).

Problem 3. (a) Convince yourself that this formal definition is the same as the more intuitive notion of a permutation as an arrangement. (b) Suppose A, B are finite sets with the same number of elements. Prove that a function $f : A \to B$ is one-to-one if and only if it is onto. In other words, f is injective if and only if f is bijective.

The number of permutations of n distinct objects is $n(n-1)(n-2)\cdots(2)(1)$. This is defined to be the *factorial* of n and denoted $n!$. It is often convenient to take the n distinct objects to simply be the numbers $1, 2, \cdots, n$. More generally, if r is a number with $0 \le r \le n$, then a *permutation of n distinct objects taken r at a time* is an arrangement of any r out of the n objects in a line.

Problem 4. Check that this is the same thing as a function $f : \{1, 2, \cdots, r\} \to S$ which is injective.

The number of permutations of n objects taken r at a time is $\frac{n!}{(n-r)!}$, and is denoted ${}^nP_r$.

Example 3. Find the number of ways of placing 8 rooks on a chessboard such that no two of the rooks can attack each other.

A chessboard is an 8×8 grid of squares, and a rook (or elephant) can move horizontally or vertically on the board to attack other pieces. So, we need to find the number of ways of choosing 8 squares on the chessboard such that there is exactly one square in each row and exactly one square in each column. Suppose we have one such configuration. We can encode this in the following table, where we have noted down the coordinates (row, column) of each rook.

Row	1	2	3	4	5	6	7	8
Col	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8

Now observe that $c_1, c_2, \cdots, c_8$ form a permutation of $1, 2, \cdots, 8$. Conversely given such c_i's, we obtain a non-attacking rook configuration. Thus, the number of required configurations is equal to the number of permutations of 8, which is $8!$. □

Problem 5. More generally, let k be an integer between 0 and 8. Find the number of non-attacking configurations of k rooks on the chessboard.

Problem 6. In the previous problem, we are assuming that the rooks are all identical. Redo the above problem under the assumption that the k rooks are of k different colours (i.e., distinguishable).

3. Combinations

The number of ways of choosing r objects from n distinct objects is denoted ${}^nC_r$. We have the formula:

$$ {}^nC_r = \frac{n!}{r!(\,(n-r)!}. $$

Example 4. Let r be a positive integer. Show that the product of r consecutive natural numbers is always divisible by $r!$.

Let us denote the r consecutive numbers (in descending order) as $n, n-1, n-2, \cdots, n-r+1$. Then their product $n(n-1)\cdots(n-r+1)$ is exactly $n!/(n-r)!$, which is just ${}^nP_r$. So, we need to show that $\frac{{}^nP_r}{r!}$ is an integer. But observe that this is just $\frac{n!}{r!\,(n-r)!} = {}^nC_r$, which is clearly an integer. □

Example 5. Consider all words (meaningful or not) that use only the letters a, b. Find the number of such words of length n which have r occurrences of a and $(n-r)$ occurrences of b.

Any such word is specified uniquely once we know the positions in which the letter a occurs. Thus, we need to pick r positions from the total available n positions. The number of possibilities is thus ${}^nC_r$. □

The previous example gives a visual proof of the binomial theorem. Let us consider two variables a, b and fully expand the product $(a+b)^n = (a+b)(a+b)\cdots(a+b)$. While expanding, let us avoid using the commutativity property, i.e., we don't allow ourselves to replace ab by ba. If we do this for small values of n, say $n = 2, 3$, we get:

$$ (a+b)(a+b) = aa + (ab+ba) + bb. $$
$$ (a+b)(a+b)(a+b) = aaa + (aab+aba+baa) + (abb+bab+bba) + bbb. $$

For general n, the corresponding right hand side will be a sum of all words of length n in a, b. Now, allowing ourselves commutativity of a, b again, we observe that the term $a^r b^{n-r}$ will occur in the expansion as many times as there are words of length n in a, b which contain r occurrences of a (and $n-r$ occurrences of b). By the preceding example, this is just ${}^nC_r$. We thus obtain the binomial theorem:

$$ (a+b)^n = \sum_{r=0}^{n} {}^nC_r\, a^r b^{n-r}. $$

Problem 7. Carry out a similar analysis with three variables a, b, c. More precisely, given non-negative integers i, j, k with $i + j + k = n$ deduce that the coefficient of $a^i b^j c^k$ in $(a+b+c)^n$ is equal to the number of words of length n in a, b, c in which a, b, c occur i, j, k times respectively. Further prove that this number equals $\frac{n!}{i!\,j!\,k!}$.

Problem 8. Generalise the above problem to k variables, for any $k \geq 2$. The resulting expansion is called the *multinomial theorem.*

We recall that the key identity satisfied by the binomial coefficients is:

$$^{n+1}C_{r+1} = {}^{n}C_r + {}^{n}C_{r+1}.$$

This is what is used to construct each row of the Pascal's triangle from the preceding row. We now show how this identity can be proved without explicitly using the formula in terms of factorials. The LHS is the number of ways of choosing $r+1$ numbers from the numbers $1, 2, \cdots, n+1$. Each such choice either contains the number $n+1$ or does not contain it. Let us count the number of choices of each type. If we are not allowed to choose $n+1$, we must then choose all our $r+1$ numbers from within the set $1, 2, \cdots, n$. The number of such ways is thus ${}^{n}C_{r+1}$. On the other hand, if we have to pick the number $n+1$, we only need to pick the remaining r numbers from amongst the numbers $1, 2, \cdots, n$. The number of choices here is ${}^{n}C_r$. The sum principle then establishes the required equality.

4. Combinations with repetitions

Problem 9. There are 4 types of flowering plants in a garden: Jasmine, Lotus, Rose, Sunflower. Assume that there are a large number of flowers of each type. We want to pluck a total of 10 flowers and put them in our basket. In how many different ways can this be done ?

Problem 10. Find the number of solutions to the equation:

$$x + y + z + w = 10,$$

where x, y, z, w must be non-negative integers.