Chennai Lectures
January 2014

Sixth problem sheet

Perverse Sheaves

1. Let Fl_n denote the complex flag variety G/B in type A_{n-1}. In other words, $\mathrm{Fl}_n = \{V^\bullet = (0 \subset V^1 \subset \ldots \subset V^n = \mathbb{C}_n) \mid \dim V^i = i\}$ is the set of flags in a fixed vector space $\mathbb{C}^n$ with basis $\{e_i\}_{1\leq i\leq n}$. There is an action of GL_n on Fl_n, and thus an action of the subgroup $S_n \subset GL_n$. The *standard flag* V_{std} is given by $V^k_{\mathrm{std}} = \mathbb{C}\cdot\langle e_i\rangle_{1\leq i\leq k}$; its stabilizer is a Borel subgroup B. For any $w \in S_n$, the dimension of $V^k_{\mathrm{std}} \cap w(V_{\mathrm{std}})^l$ is equal to the size of the intersection $\{1, 2, \ldots, k\} \cap \{w(1), w(2), \ldots, w(l)\}$. For any two flags $V^\bullet$ and $W^\bullet$, we say that they are in *relative position* w if $\dim(V^k \cap W^l) = \dim(V^k_{\mathrm{std}} \cap w(V_{\mathrm{std}})^l)$.

a) Show that Fl_n splits into B orbits based on the relative position of a flag with the standard flag, and that this agrees with the usual Bruhat decomposition of G/B. Show that $\mathrm{Fl}_n \times \mathrm{Fl}_n$ splits into G orbits based on the relative position of the two flags. Show that the orbit closure relation agrees with the Bruhat order, in either setting.

Clearly $V^\bullet$ and $W^\bullet$ are in relative position $s_i \in S \subset S_n$ if and only if $V^i \neq W^i$ and $V^k = W^k$ for all $k \neq i$. We say that $V^\bullet$ and $W^\bullet$ are in *relative position* $\overline{s_i}$ if $V^k = W^k$ for all $k \neq i$ (with no condition on V^i and W^i). Let $\underline{w} = s_{i_1}s_{i_2}\ldots s_{i_d}$ be a sequence of simple reflections. The *Bott-Samelson resolution* $BS(\underline{w})$ is the space consisting of sequences of flags, ending in the standard flag, and successively in relative position determined by $\underline{w}$:

$$\{(V_i^\bullet)_{i=0}^d \mid V_d^\bullet = V^\bullet_{\mathrm{std}}, \text{ and the pair } (V^\bullet_{k-1}, V^\bullet_k) \text{ is in relative position } \overline{s_{i_k}} \text{ for each } 1 \leq k \leq d\}.$$

It is equipped with a map $\mu\colon BS(\underline{w}) \to \mathrm{Fl}_n$, $\mu((V_i^\bullet)) = V_0^\bullet$.

b) Show that this description of the Bott-Samelson resolution agrees with the one given in lecture.

Set $n = 4$, and let s, t, u denote the simple reflections in S_4 with $su = us$. For an arbitrary flag $W^\bullet$ in each orbit, calculate the fiber $\mu^{-1}(W^\bullet)$ when:

c) $\underline{w} = tt$.

d) $\underline{w} = sts$.

e) $\underline{w} = tsut$.

f) $\underline{w} = sutsu$.

Now, for each of the above cases, construct the table for $\mu_*(\mathbb{C}[\ell(\underline{w})])$. Use these tables (and possibly other calculations) to decompose this pushforward into $\mathcal{I}$ sheaves.

2. This is a family of possible exercises, imitating a computation from lecture. Consider a partition λ of n, and let P_λ be the corresponding parabolic subgroup of GL_n. We will consider P_λ acting on all Grassmannians $\mathbb{G}(k, n)$ for $0 \leq k \leq n$.

a) Classify the P_λ orbits on $\mathbb{G}(k, n)$, and prove that your classification is correct. Compute their dimensions.

b) Find a resolution of singularities of each orbit closure. Compute the fibers over each orbit in this resolution.

c) Construct a table for the pushforward of the IC sheaf (i.e. constant sheaf with shift) on each resolution of singularities. Use this to compute the IC sheaf of the orbit.

Now consider the partial flag variety $\mathbb{F}(k, k+1, n)$, with its forgetful maps $p\colon \mathbb{F}(k, k+1, n) \to \mathbb{G}(k, n)$ and $q\colon \mathbb{F}(k, k+1, n) \to \mathbb{G}(k+1, n)$. Let d_k be the dimension of $\mathbb{G}(k, n)$. Define E to be the functor $q_* p^*(\cdot)[d_{k+1} - d_k]$ from perverse sheaves with shifts on $\mathbb{G}(k, n)$ to perverse sheaves with shifts on $\mathbb{G}(k+1, n)$. Let F be the functor $p_* q^*(\cdot)[d_k - d_{k+1}]$ in the other direction.

d) Compute the table of E and F applied to each IC sheaf. Compute the decomposition into perverse sheaves.

e) Verify that, on the Grothendieck group, $[E]$ and $[F]$ induce an action of $U_v(\mathfrak{sl}_2)$, giving the representation $V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots \otimes V_{\lambda_k}$.

f) Find a subcategory of each category of perverse sheaves, preserved under the functors E and F, whose Grothendieck group gives the subrepresentation V_λ. Are there subcategories for other subrepresentations?

More theory of Soergel bimodules and Lefschetz operators

3. Prove that the summand $B_x \subset BS(\underline{x})$ (for a reduced expression) contains both c_{bot} and c_{top}.

4. We continue the notation of Exercise 9 from the Fourth problem sheet. Assume that B is equipped with a Lefschetz operator L given by left multiplication, so that $\overline{B}$ has hard Lefschetz. Let $\{e_i\}$ be a collection of elements of B^{-k-1} which project to an orthonormal basis of $\overline{B}^{-k-1}$ (with respect to the Lefschetz form). Let $\{f_i\}$ be a collection of elements projecting to an orthonormal basis of $P^{-k+1} \subset \overline{B}^{-k+1}$.

a) Find a basis for $\overline{B}^{-k+1}$. Find a basis for $\overline{BB_s}^{-k}$, using the maps α and β.

Let L continue to denote the same operator of left multiplication, now considered to act on BB_s. Let M denote multiplication by some linear polynomial ρ immediately to the left of B_s in BB_s, and suppose that $\partial_s(\rho) = 1$. For $v, w \in \overline{BB_s}^{-k}$, one can pair them by the formula $\langle v, L^{k-1} M w\rangle$. As discussed in lecture, this pairing is the limit of the Lefschetz pairing induced by $L + \zeta M$ on $\overline{BB_s}$.

b) Compute the matrix of this pairing in the basis you found above.

c) Deduce that the signature of this pairing on $\overline{BB_s}^{-k}$ is equal to the signature of the Lefschetz form on $P^{-k+1} \subset \overline{B}^{-k+1}$.

Rouquier complexes

5. Let F_s and F_s^{-1} denote the Rouquier complexes introduced in lectures. Check that $F_s F_s^{-1} \cong R$ in $K^b(R\text{-Bim})$ as sketched in lectures.

6. Compute the minimal complex of $F_s^{\otimes m}$ for $m \geq 0$. Describe its perverse filtration explicitly.

7. Write down the summands appearing in the minimal complex of $F_s F_u F_t F_s F_u$.

8. Suppose that $m_{st} = 2$. Find explicitly a chain map from $F_s F_t$ to $F_t F_s$ and back. Renormalize your maps such that the composition is the identity chain map.

9. This exercise is very very computational! (Hint: If you're stuck, look at a paper by Elias-Krasner.) Suppose that $m_{st} = 3$. Find the most general chain map (of degree 0) from $F_s F_t F_s$ to $F_t F_s F_t$ and vice versa (i.e. you should get families of maps given by certain parameters). Compute their composition, an endomorphism of $F_s F_t F_s$. For certain parameters, find a homotopy map between this composition and the identity chain map.