Chennai Lectures

January 2014

Fifth problem sheet

Duality and invariant forms

1. Given an R-bimodule B, its dual $\mathbb{D}B$ is defined to be $\mathrm{Hom}_{(-,R)}(B,R)$, the right R-module maps. Clearly $\mathbb{D}^2 = \mathbb{1}$ on any bimodule which is free as a right R-module.

a) Show that $\mathrm{Hom}^0_{(R,R)}(B,\mathbb{D}B)$ is isomorphic to the space of invariant forms on B. If a map $B \to \mathbb{D}B$ is an isomorphism, what does this say about the corresponding invariant form?

b) What is $\mathbb{D}B_s$? What about $\mathbb{D}BS(\underline{w})$?

c) Show (by definition) that $\mathbb{D}B_w \cong B_w$ for all $w \in W$, and therefore there exists a nondegenerate invariant form on B_w.

d) If the Soergel conjecture holds, show that any non-zero invariant form on B_w is nondegenerate. Show that the global intersection form on $BS(\underline{w})$ for a reduced expression restricts to a nonzero form on B_w.

Lefschetz linear algebra

2. This question explores a combinatorial description of the cohomology ring of the Grassmannian $Gr(3,6)$. Let $H^\bullet$ denote the following graded vector space. It has a basis given by partitions λ which fit in a 3×3 square. The original degree of λ is twice the number of boxes in λ; then we renormalize the degrees so that $H^\bullet$ is centered around degree 0. There is a pairing on $H^\bullet$ which satisfies $\langle \lambda, \mu\rangle = 0$ unless one can glue λ and the 180 degree rotation of μ into a 3×3 square, in which case $\langle \lambda, \mu\rangle = 1$.

a) Let L denote the operator of degree 2 which sends λ to the sum of all ways to add a box. Show that L is a Lefschetz operator.

b) Show that L has (hL). (What is the least amount of work one needs to do to check this?)

c) Show that L has (HR).

3. Let $H = \bigoplus H^i$ be a finite dimensional graded $\mathbb{R}$-vector space and $L : H^\bullet \to H^{\bullet+2}$ an operator of degree 2. Show that H admits a representation of $\mathfrak{sl}_2(\mathbb{R}) = \mathbb{R}f \oplus \mathbb{R}h \oplus \mathbb{R}e$ with $e = L$ and $hx = mx$ for all $x \in H^m$ if and only if L satisfies the hard Lefschetz theorem (i.e. $L^m : H^{-m} \to H^m$ is an isomorphism for all $m \geq 0$).

4. Suppose that $H = \oplus H^i$ and $W = \oplus W^j$ are finite dimensional graded real vector spaces with forms $\langle -,-\rangle$ and Lefschetz operators L_H and L_W. Suppose that $H^{\mathrm{odd}} = 0$ or $H^{\mathrm{even}} = 0$, that L satisfies the hard Lefschetz theorem on H and that

$$\underline{\dim} W := \sum \dim W^i v^i = (v + v^{-1})\underline{\dim} H.$$

Show that W satisfies (HR) if and only if the signature of the Lefschetz form $(-,-)^{-i}_{L_W}$ on W^{-i} is equal to the dimension of the primitive subspace $P^{-i+1}_{L_H} \subset H^{-i+1}$ (by convention $P^1_{L_H} = 0$).

5. Let $\sigma\colon V^\bullet \to W^\bullet$ be a map of degree $+1$, commuting with certain Lefschetz operators on either side, and satisfying $(v,v')^{-i}_{L_V} = (\sigma v, \sigma v')^{-i+1}_{L_W}$. Suppose that σ is injective from negative degrees. Find some reasonable conditions (similar to those coming from the weak Lefschetz theorem) which would imply that σ maps $P^{-i}_{L_V}$ to $P^{-i+1}_{L_W}$ for $i > 0$.

6. Let $\phi\colon V^\bullet \to W^\bullet$ be a map of degree d, commuting with certain Lefschetz operators on either side, and suppose that V has (hL). When $d < 0$, show that the restriction of $\langle\ ,\ \rangle_W$ to the image of V is identically zero. What happens if $d > 0$?

7. Consider the two inclusion maps $i_1, i_2\colon B_s \to B_sB_s$, where i_1 has degree -1 and i_2 has degree $+1$ (i_2 is not canonical, but choose the one from class). Show that the intersection form on B_sB_s restricts to zero on the image of i_1, but does not restrict to zero on the image of i_2.

Lefschetz calculations in Soergel bimodules

8. Consider $\overline{(B_sB_s)}$, with the Lefschetz operator

$$L_{a,b} := (a\rho\cdot -)\,\mathrm{id}_{B_s} + \mathrm{id}_{B_s}(b\rho\cdot -)$$

for some $a, b \in \mathbb{R}$. For which a, b does the hard Lefschetz property hold? For which a, b do the Hodge-Riemann bilinear relations hold? For which a, b does (HR) hold with the opposite signatures?

9. In this exercise we prove an "easy" case of hard Lefschetz. Assume that B_x is a Soergel bimodule such that hard Lefschetz holds on $\overline{B_x}$. Consider the operator

$$L_\zeta := (\rho\cdot -)\,\mathrm{id}_{B_s} + \mathrm{id}_{B_x}(\zeta\rho\cdot -)$$

on B_xB_s. It induces a Lefschetz operator L_ζ on $\overline{B_xB_s}$. (You can equip B_x with an invariant form if you wish, but it won't be important for this exercise.)

a) Let $s \in S$ be such that $xs < s$. Show that $B_xB_s = B_x(1) \oplus B_x(-1)$. (You should be able to give an abstract argument, but in part b) the following fact is useful (see "Singular Soergel bimodules"): there exists an (R, R^s)-bimodule $B_{\overline{x}}$ such that $B_{\overline{x}} \otimes_{R^s} R \cong B_x$.)

b) Rewrite the Lefschetz operator L_z on B_xB_s using a fixed choice of isomorphism $B_xB_s = B_x(1) \oplus B_x(-1)$. Conclude that in the right quotient $\overline{B_xB_s}$, L_ζ has the form

$$\begin{pmatrix} \rho\cdot - & 0 \\ \zeta\gamma & \rho\cdot - \end{pmatrix}.$$

for some non-zero scalar γ. (As above, $\rho\cdot -$ denotes the degree two endomorphism of left multiplication by ρ.)

c) Conclude that L_ζ satisfies hard Lefschetz on $\overline{B_xB_s}$ if and only if $\zeta \neq 0$.

10. In this lecture series, we have been assuming that $a_{s,t} = -2\cos\frac{\pi}{m_{st}}$, or in other words, that $a_{s,t} = -(q + q^{-1})$ where $q = e^{\pm\frac{\pi i}{m}}$ is a primitive $2m$-th root of unity. In previous exercises, we have seen that one can set q to be other primitive $2m$-th roots of unity and still obtain an action of W. What positivity considerations will fail if q is set to one of these other roots of unity? For example, what if $m = 53$ and $q = e^{\frac{3\pi i}{53}}$? (Hint: Consider Exercise 10(c) from the Fourth Problem Sheet.)