Chennai Lectures

January 2014

Fourth problem sheet

Light leaves and the local intersection form

1. Describe all light leaves maps from $ss\dots s$ (k times). Compute the local intersection form in each degree. Does it confirm your expectations?

2. Continue the setup of Exercise 9 from the first problem sheet (type D_4). Draw all the light leaves maps from $suvtsuv$ with target suv. Compute the local intersection form in each degree. In which characteristics does one expect the behavior of the decomposition of $BS(suvtsuv)$ to change?

3. Here we give a torsion example that surprised a few people last decade! Consider S_8, a Weyl group of type A_7. Let $i = (i, i+1)$ (we write i instead of s_i for reasons that should become clear).

a) Consider the reduced expression:

$$\underline{w} = 1357246352461357.$$

Show that $\mathbf{e} = 1111010110100000$ is the unique subexpression of defect zero with endpoint

$$w_I = 13435437.$$

(Note that w_I is the maximal element in the standard parabolic subgroup generated by $I = \{1, 3, 4, 5, 7\}$.) Draw the corresponding light leaf.

b) Hence calculate the local intersection form in degree zero of $\underline{w}$ at w_I. (A lengthy computation, but a reasonable one.)

4. After localization to Q, the fraction field of R, the Bott-Samelson bimodule $B_s \otimes_R Q$ splits as a direct sum of Q_s and Q (when using localization we ignore the grading). Therefore, for any subsequence $\mathbf{e} \subset \underline{w}$, there is a summand $Q_{\mathbf{e}} \overset{\oplus}{\subset} BS(\underline{w}) \otimes_R Q$, a tensor product of either $Q_{\underline{w}_i}$ or Q depending on whether $\mathbf{e}_i$ is 1 or 0. Obviously $Q_{\mathbf{e}} \cong Q_x$ when $\mathbf{e}$ expresses the element x.

Use localization and the Bruhat path dominance order to prove that the images in $\mathbb{BS}\mathrm{Bim}$ of the light leaves maps in $\mathbb{LL}_{\underline{w},x}$ are all linearly independent.

5. Show that the functor from $\mathcal{D}$ to $\mathbb{BS}\mathrm{Bim}$ is an equivalence of categories, assuming that double leaves form a basis for morphisms in $\mathcal{D}$.

Global intersection forms and Bott-Samelson calculations

6. Recall that B_s has a basis as a right R-module given by $c_1 = c_{bot} = 1 \otimes 1$ and $c_s = \frac{\alpha_s}{2} \otimes 1 + 1 \otimes \frac{\alpha_s}{2}$. Let $f \in R$ be arbitrary. Compute fc_1 and fc_s in this basis. Now recall that $BS(\underline{w})$ has a basis given by $c_{\underline{\varepsilon}}$. For a linear polynomial f, express $fc_{\underline{\varepsilon}}$ in this basis.

7. For a Soergel bimodule B, let $\overline{B}$ denote $B \otimes_R \mathbb{R}$ be the *right quotient*. For example, $\overline{BS(\underline{w})}$ has a basis over $\mathbb{R}$ given by 01-sequences. Just as $BS(\underline{w})$ has an intersection form valued in R, so too does $\overline{BS(\underline{w})}$ have an intersection form valued in $\mathbb{R}$.

The endomorphism  of B_sB_s gives a degree 2 endomorphism L of the vector space $\overline{B_sB_s}$. What is $\langle c_{\mathrm{bot}}, L^2(c_{\mathrm{bot}})\rangle$? What is $\langle L(c_{\mathrm{bot}}), L(c_{\mathrm{bot}})\rangle$? Find an element b of degree zero which is perpendicular to $L(c_{\mathrm{bot}})$. What is $\langle b, b\rangle$?

Now let L_0 be the degree 2 endomorphism of $\overline{B_sB_s}$ given by left multiplication by α_s. What is $L_0^2(c_{\mathrm{bot}})$?

8. In the previous question we defined the intersection form on $\overline{BS(\underline{w})}$. Now we repeat some of the same calculations with $B_sB_tB_s$ when $m_{st} = 3$. Let $\rho \in \mathfrak{h}^*$ satisfy $\partial_s(\rho) = \partial_t(\rho) = 1$. Let L be the degree 2 endomorphism of $\overline{B_sB_tB_s}$ given by left multiplication by ρ.

What is $L^3(c_{\text{bot}})$? What is $\langle c_{\text{bot}}, L^3(c_{\text{bot}})\rangle$? Find a basis for $\overline{B_sB_tB_s}^{-1}$ (i.e. the elements in degree -1) in the kernel of L^2. Are they orthogonal to $L^2(c_{\text{bot}})$ under the intersection form? Show that the form $(v, w) = \langle v, Lw\rangle$ on this orthogonal subspace of $\overline{B_sB_tB_s}^{-1}$ is negative definite.

Bonus problem: what does the picture look like when restricted to the summand $B_s \overset{\oplus}{\subset} B_sB_tB_s$? What does it look like when restricted to the summand $B_{sts} \overset{\oplus}{\subset} B_sB_tB_s$?

9. Fix a Soergel bimodule B and consider the two maps $\alpha, \beta : B \to BB_s = B \otimes_R B_s$ given by

$$\alpha(b) := bc_{\text{id}} \quad \text{and} \quad \beta(b) := bc_s.$$

Together, $\alpha(B)$ and $\beta(B)$ span BB_s. Show that β is a morphism of bimodules, whilst α is a morphism of left modules. Find a formula for $\alpha(br)$ for $b \in B$ and $r \in R$.

Suppose that B is equipped with an invariant form $\langle -, -\rangle_B$. Prove that there is a unique invariant form $\langle -, -\rangle_{BB_s}$ on BB_s, which we call the *induced form*, satisfying

$$\langle \alpha(b), \alpha(b')\rangle_{BB_s} = \partial_s(\langle b, b'\rangle_B) \tag{1}$$
$$\langle \alpha(b), \beta(b')\rangle_{BB_s} = \langle b, b'\rangle_B \text{ and } \langle \beta(b), \alpha(b')\rangle_{BB_s} = \langle b, b'\rangle_B \tag{2}$$
$$\langle \beta(b), \beta(b')\rangle_{BB_s} = \langle b, b'\rangle_B \alpha_s \tag{3}$$

for all $b, b' \in B$. Show that the intersection form on a Bott-Samelson bimodule agrees with the form induced many times from the canonical form on R.

Now consider $\overline{BB_s}$, with its induced form valued in $\mathbb{R}$. Calculate a matrix for this form in some basis. Prove that the induced form is non-degenerate whenever the original form on $\overline{B}$ is non-degenerate.

10. In lectures we saw that for any expression $\underline{w}$, $BS(\underline{w})$ has a basis as a right R-module given by 01-sequences. It contains two canonical elements c_{bot} and c_{top} which project to elements of minimal and maximal degree in $\overline{BS(\underline{w})}$. In this exercise we find a recursive formula for

$$N_{\underline{w}}(f) := \langle f^{\ell(\underline{w})} c_{\text{bot}}, c_{\text{bot}}\rangle.$$

for any degree two element $f \in R$.

a) Find a formula for $N_{\underline{w}}(f)$ in terms of $N_{\underline{w}'}(f)$, over all subexpressions $\underline{w}'$ obtained by omitting a simple reflection from $\underline{w}$.

b) Show that $N_{\underline{w}}(f) = 0$ unless $\underline{w}$ is reduced. (*Hint:* It might help to use the light leaves description of $BS(\underline{w})$ or the decomposition of $BS(\underline{w})$ into indecomposable Soergel bimodules.) Use this to simplify your formula in part (a).

c) Suppose that $\partial_s(f) > 0$ for all $s \in S$. Show that $N_{\underline{w}}(f) > 0$ for $\underline{w}$ reduced. (First prove that $sw > w$ if and only if $\partial_s(wf) > 0$.)

Krull-Schmidt categories

Recall that a *Krull-Schmidt category* is an additive category in which every object is isomorphic to a finite direct sum of indecomposable objects, and an object is indecomposable if and only if its endomorphism ring is local.

11. Some exercises to get used to Krull-Schmidt categories:

a) Show that the Krull-Schmidt theorem holds in Krull-Schmidt categories: any object can be written as a direct sum of indecomposable objects, and this decomposition is unique up to permutation of the factors.

b) *(Idempotent lifting)* Let A be an algebra and $\mathfrak{m} \subset A$ an ideal such that $\mathfrak{m}^2 = 0$. Show that given an idempotent $e \in A/\mathfrak{m}$ there exists an idempotent $\widetilde{e} \in A$ such that $e = \widetilde{e}$ in $A/\mathfrak{m}$. Now prove the same statement assuming only that A is complete with respect to the topology defined by $\mathfrak{m}$.

c) Let $(\mathbb{O}, \mathfrak{m})$ be a complete local ring. Let $\mathcal{C}$ be a Karoubian $\mathbb{O}$-linear additive category such that all hom spaces are finitely generated. Show that $\mathcal{C}$ is Krull-Schmidt. (*Hint:* It might help to first consider the case when $\mathbb{O}$ is a field.)

d) Show that the category of graded modules over a polynomial ring is a Krull-Schmidt category. Conclude that the category of Soergel bimodules is Krull-Schmidt.

e) (If you're up for it) Let X be an affine variety. When does the Krull-Schmidt theorem hold for vector bundles on X? (Answer: almost never.) Conclude that the Krull-Schmidt theorem fails for ungraded modules over a polynomial ring. (Optional: show that the Krull-Schmidt theorem holds for vector bundles on a projective algebraic variety.)

12. Let $\mathcal{C}$ be a Krull-Schmidt category over an algebraically closed field $\mathbb{k}$. Show that the multiplicity of B as summand of X is given by the rank of the form

$$\mathrm{Hom}(B, X) \times \mathrm{Hom}(X, B) \to \mathrm{End}(B)/\mathfrak{m}_B.$$

where $\mathfrak{m}_B$ denotes the maximal ideal of $\mathrm{End}(B)$. What is the correct statement for general fields or local rings $\mathbb{k}$?