Chennai Lectures
January 2014

Third problem sheet

Diagrammatics for Frobenius extensions and Bott-Samelson bimodules

1. For any Frobenius extension $A \subset B$, show diagrammatically that $B \otimes_A B$ is a Frobenius algebra object in the category of B-bimodules. (Hint: Your answer should involve a deformation retract.)

2. (*) Check that the one color relations hold in Soergel bimodules. Verify the decomposition $B_s B_s \cong B_s(1) \oplus B_s(-1)$.

3. (*) In the previous exercise sheet, you calculated the graded rank (as a free right R-module) of

a) $\mathrm{Hom}(B_s, B_t)$,

b) $\mathrm{Hom}(B_s, B_s B_t B_s)$ when $m_{st} > 2$,

c) $\mathrm{Hom}(B_s B_s, B_s)$.

Now construct diagrammatic bases for these spaces. (Hint: They only use maps you already know how to draw.)

Constructing idempotents

4. (*) In this exercise, we work in type B_2, so that $m_{st} = 4$. One has $a_{s,t} = -1$ and $a_{t,s} = -2$.

a) In type B_2, write $\underline{H}_s \underline{H}_t \underline{H}_s \underline{H}_t$ as a sum of KL basis elements. How do you expect $B_s B_t B_s B_t$ to decompose?

b) Calculate the graded rank of $\mathrm{Hom}(B_s B_t, B_s B_t B_s B_t)$. Compute a diagrammatic basis of maps in degree 0 (you should have found it to be a 2-dimensional space).

c) Calculate the graded rank of $\mathrm{Hom}(B_s B_t B_s B_t, B_s B_t)$. Compute a diagrammatic basis of maps in degree 0. Why is this really easy, given the last part?

d) Calculate the graded rank of $\mathrm{End}(B_s B_t)$ and deduce that the only degree zero map is the identity.

e) Therefore, one can construct a 2×2 matrix given by composing a map $B_s B_t \to B_s B_t B_s B_t$ of degree 0 with a map $B_s B_t B_s B_t \to B_s B_t$ of degree 0, and computing the coefficient of the identity. This is called a *local intersection form*; one thinks of it as a bilinear form on $\mathrm{Hom}(B_s B_t B_s B_t, B_s B_t)$... How? Compute this matrix.

f) Whenever two maps pair under the local intersection form to the value 1, one can construct an idempotent in $\mathrm{End}(B_s B_t B_s B_t)$ which factors through $B_s B_t$. Whenever one has dual bases under the local intersection form, the corresponding idempotents will be orthogonal. Find dual bases and compute these orthogonal idempotents.

g) You have just proven that $B_s B_t$ occurs as a summand inside $B_s B_t B_s B_t$ precisely n times. Can there be any other summands besides B_{stst}? Why or why not?

h) Suppose that we work in characteristic 2. How many times does $B_s B_t$ occurs as a summand inside $B_s B_t B_s B_t$?

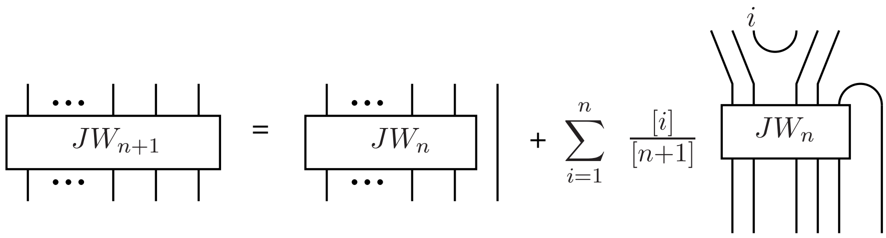
5. What happens if you repeat Q4 in type H_2? One has $m_{st} = 5$, and $a_{s,t} = a_{t,s} = -\phi$, the (negative) golden ratio.

6. If you want more exercise, repeat Q4 in type H_2, except with the goal of decomposing $B_s B_t B_s B_t B_s$.

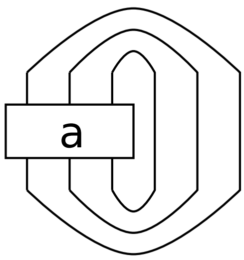
Temperley-Lieb algebra

7. (*) Let TL_n be the Temperley-Lieb algebra with n strands, where a circle evaluates to $-[2] = -(q + q^{-1}) \in \mathbb{Z}[q, q^{-1}]$. Now base change to $\mathbb{Q}(q)$. Show that the space of all elements killed by caps above (resp. cups below) is one-dimensional, and show that these spaces agree.

The Jones-Wenzl projector $JW_n \in TL_n$ is uniquely specified in this one-dimensional kernel by the fact that the coefficient of the identity is 1. Verify the following recursive formula.



The *trace* of an element $a \in TL_n$ is the evaluation in $\mathbb{Z}[q, q^{-1}]$ of the closed diagram below. Calculate the trace of JW_n (hint: use induction). In a specialization of $\mathbb{Z}[q, q^{-1}]$ where the trace of JW_n is zero, what do you get when you rotate JW_n by one strand?



8. Compute the decomposition of $V \otimes V \otimes V$ into direct summands, by constructing an idempotent decomposition of the identity. Does this remind you of any previous exercises? What happens when q is an 8-th root of unity?

Two-color and three-color relations - note: didn't cover this in lecture today

9. Let $m_{st} = m < \infty$. For $k > 0$, let $\underline{w} = stst \ldots st$ of length $2(m + k)$. What is the dimension of $\mathrm{Hom}(BS(\underline{w}), R)$ in degree $-2k$? Draw several different graphs realizing the same morphism in this space. Start with $k = 1$, of course.

10. Let $S = \{s, t, u\}$ be type A_3. Let $\underline{w} = tstuts$ and let $\underline{y} = utstut$ be two expressions for the longest element $w_0 \in W$. There are (essentially) two paths from $\underline{w}$ to $\underline{y}$ in the reduced expression graph of w_0. Find a reasonably quick proof that the two corresponding morphisms of Bott-Samelson bimodules are not equal.

11. Verify that the "dot relation" and the "associativity relation" imply the "idempotent relation."

12. Using the reduced expression graphs you draw for Exercise sheet 1, draw the Zamolodzhikov relations for all finite rank 3 Coxeter groups (at least, modulo lower terms).

Ideas for more exercises

Learn about Frobenius hypercubes in arXiv:1308.5994 and verify some of the relations.

Learn about $\mathfrak{sl}_n$-webs in arXiv:1210:6437 and try to compute some idempotent decompositions for $\mathfrak{sl}_3$ tensor products.