ECE15: Linear Algebra Problem Set

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Jordan Canonical Form (JCF)

1. For a real 8×8 matrix with characteristic polynomial $(t-1)^3(t+1)^5$, what are the different possible values of its minimal polynomial?

   (a) 8 (b) 12 (c) 15 (d) 21

2. On a real finite dimensional vector space, let T be a linear operator with characteristic polynomial $(t-1)^4(t-2)^3(t-3)^2(t-4)$. What is the dimension of the image of the operator $(T-1)^4$?

   (a) 7 (b) 6 (c) 5 (d) 4

3. Which of the following is the JCF of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$?

   (a) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

4. How many different similarity classes are there of real 8×8 matrices with characteristic polynomial $(t-2)^2(t-3)^4(t-4)^2$?

   (a) 8 (b) 16 (c) 20 (d) 24

5. Let T be a nilpotent operator on a 5-dimensional vector space. Which of the following is NOT a possible value for the dimension of the kernel of T^3?

   (a) 2 (b) 3 (c) 4 (d) 5

Orthogonal projections

In the sequel, A^t represents the transpose of the matrix A.

1. Find the matrix P that projects every vector in $\mathbb{R}^3$ onto the line in the direction of $(2,1,3)^t$. What are the column space and null space of P? Find a basis for each space. What are the eigenvalues of P? What is the trace of P?

2. Suppose that a real $r \times c$ matrix A has linearly independent columns (so that $r \geq c$). Show that A^tA is invertible. Is the assertion true if real is replaced by complex? What is the correct complex analogue of the assertion?

3. Suppose that a real $r \times c$ matrix A has linearly independent columns (so that $r \geq c$). Prove that $A(A^tA)^{-1}A^t$ is the matrix representing the orthogonal projection to the column space of A. (It follows from the previous item that A^tA is invertible.)

4. Let q_1 and q_2 be orthonormal vectors in $\mathbb{R}^5$. Give a formula for the projection p of any vector b onto the plane spanned by q_1 and q_2 (express p as a linear combination of q_1 and q_2). (Apply the formula in the previous item.)

Miscellaneous

1. (Choose the right option) Suppose that a 3×5 matrix A has rank 3. Then the equation $Ax = b$ (always / sometimes but not always) has (a unique solution / many solutions / no solution). What is the column space of A? What is the dimension of the null space of A?

2. $$A = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{pmatrix}$$

   Find bases for the null space and column space of A. What is the row reduced echelon form of the 6×8 matrix
   $$\begin{pmatrix} A & A \\ A & A \end{pmatrix}$$

3. Explain in words how knowing all solutions to the equation $Ax = b$ decides whether or not b is in the column space of A.

4. Let A_n denote the $n \times n$ matrix all of whose non-diagonal entries are -1 and all of whose diagonal entries are 0. What is $\det A_n$? What is the entry in position $(1,1)$ of the inverse of A? Find the eigenvalues, their multiplicities, and corresponding eigenvectors for A_n.