THE CAKRAVĀLA METHOD FOR SOLVING QUADRATIC INDETERMINATE EQUATIONS

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OUTLINE

- The *vargaprakṛti* equation $X^2 – D Y^2 = K$, and Brahmagupta's *bhāvanā* process (c. 628 CE).
- The *cakravāla* method of solution of Jayadeva (c. 10th cent) and Bhāskara (c.1150).
- Bhāskara's examples $X^2 – 61Y^2 = 1$, $X^2 – 67Y^2 = 1$.
- Analysis of the *cakravāla* method by Krishnaswami Ayyangar.
- History of the so called "Pell's Equation" $X^2 - D Y^2 = 1$.
- Solution of "Pell's equation" by expansion of $\sqrt{D}$ into a simple continued fraction (c. 18th cent).
- Bhāskara semi-regular continued fraction expansion of $\sqrt{D}$
- Optimality of the *cakravāla* method.

VARGA PRAKṚTI

In Chapter XVIII of his *Brāhmasphuṭa-siddhānta* (c.628 CE), Brahmagupta considers the problem of solving for **integral values** of X, Y, the equation

$$\mathbf{X^2 - D\,Y^2 = K,}$$

given a non-square integer $D > 0$, and an integer K.

X is called the larger root (*jyeṣṭha-mūla*), Y is called the smaller root (*kaniṣṭha-mūla*), D is the *prakṛti*, K is the *kṣepa*.

One motivation for this problem is that of finding rational approximations to square-root of D. If X, Y are integers such that $X^2 - D\,Y^2 = 1$, then,

$$\left|\sqrt{D} - \frac{x}{y}\right| \le \frac{1}{2xy},$$

The *Śulva-sūtra* approximation (prior to 800 BCE):

$\sqrt{2} \approx 1 + 1/3 + 1/3.4 - 1/3.4.34 = 577/408$ $\qquad [(577)^2 - 2\,(408)^2 = 1]$

BRAHMAGUPTA'S BHĀVANĀ

मूलं द्विधेष्टवर्गाद् गुणकगुणादिष्टयुतविहीनाच्च।
आद्यवधो गुणकगुणः सहान्त्यघातेन कृतमन्त्यम्॥
वज्रवधैक्यं प्रथमं प्रक्षेपःक्षेपवधतुल्यः।
प्रक्षेपशोधकहृते मूले प्रक्षेपके रूपे॥

If $X_1^2 - D\,Y_1^2 = K_1$ and $X_2^2 - D\,Y_2^2 = K_2$ then

$$\mathbf{(X_1 X_2 \pm D\,Y_1 Y_2)^2 - D\,(X_1 Y_2 \pm X_2 Y_1)^2 = K_1 K_2}$$

In particular given X^2 -D $Y^2 = K$, we get the rational solution

$$[(X^2 + D\,Y^2)/K]^2 - D\,[(2XY)/K]^2 = 1$$

Also, if one solution of the Equation X^2- D $Y^2 = 1$ is found, an infinite number of solutions can be found, via

$$(X, Y) \rightarrow (X^2 + D\,Y^2, 2XY)$$

USE OF BHĀVANĀ WHEN K = -1, ±2, ±4

The *bhāvanā* principle can be use to obtain a solution of the equation

$$X^2 - D\,Y^2 = 1,$$

if we have a solution of the equation

$$X_1^{\,2} - D\,Y_1^{\,2} = K, \text{ for } K = -1, \pm 2, \pm 4.$$

$$
\begin{aligned}
K = -1\ : x &= x_1^2 + Dy_1^2,\ y = 2x_1y_1. \\
K = \pm 2 : x &= \frac{(x_1^2 + Dy_1^2)}{2},\ y = x_1y_1. \\
K = -4 : x &= (x_1^2 + 2)\left[\frac{1}{2}(x_1^2 + 1)(x_1^2 + 3) - 1\right], \\
y &= \frac{x_1y_1(x_1^2 + 1)(x_1^2 + 3)}{2}. \\
K = 4 : x &= \frac{(x_1^2 - 2)}{2},\ y = \frac{x_1y_1}{2},\ \text{if } x_1 \text{ is even}, \\
x &= \frac{x_1(x_1^2 - 3)}{2},\ y = \frac{y_1(x_1^2 - 1)}{2},\ \text{if } x_1 \text{ is odd}.
\end{aligned}
$$

BRAHMAGUPTA’S EXAMPLES

राशिकलाशेषकृतिं द्विनवतिगुणितां त्र्यशीतिगुणितां वा।
सैकां ज्ञदिने वर्गं कुर्वन्नावत्सराद्गणकः॥

To solve, **$X^2 - D\,Y^2 = 1$, for D = 92, 83:**

$10^2 - 92.1^2 = 8$

Doing the *bhāvanā* of the above with itself,

$192^2 - 92.20^2 = 64$ $[10^2 + 92.1^2 = 192$ and $2.10.1 = 20]$

Dividing both sides by 64,

$24^2 - 92.(5/2)^2 = 1$

Doing the *bhāvanā* of the above with itself,

$\mathbf{1151^2 - 92.120^2 = 1}$ $[24^2 + 92.(5/2)^2 = 1151$ and $2.24.(5/2) = 120]$

Similarly,

$9^2 - 83.1^2 = -2$

Doing the *bhāvanā* of the above with itself,

$164^2 - 83.18^2 = 4$ and hence, $\mathbf{82^2 - 83.9^2 = 1.}$

CAKRAVĀLA: THE CYCLIC METHOD

The first known description of the *Cakravāla* or the Cyclic Method occurs in a work of Udayadivākara (c.1073), who cites the verses of Ācārya Jayadeva.

In his *Bījagaṇita*, Bhāskarācārya (c.1150) has given the following description of the *cakravāla* method:

ह्रस्वज्येष्ठपदक्षेपान् भाज्यप्रक्षेपभाजकान्। कृत्वा कल्प्यो गुणस्तत्र तथा प्रकृतितश्च्युते॥
गुणवर्गे प्रकृत्योनेऽथवाल्पं शेषकं यथा। तत्तु क्षेपहृतं क्षेपो व्यस्तः प्रकृतितश्च्युते॥
गुणलब्धिः पदं ह्रस्वं ततो ज्येष्ठमतोऽसकृत्। त्यक्त्वा पूर्वपदक्षेपांश्चक्रवालमिदं जगुः॥
चतुर्द्वेकयुतावेवमभिन्ने भवतः पदे। चतुर्द्विक्षेपमूलाभ्यां रूपक्षेपार्थभावना॥

THE CAKRAVĀLA METHOD

Given X_i, Y_i, K_i such that $X_i^{\ 2} - D\,Y_i^{\ 2} = K_i$

First find P_{i+1} as follows:

(I) Use *kuṭṭaka* process to solve (the linear indeterminate equation)

$$\mathbf{(Y_i\,P_{i+1} + X_i)/\,|K_i| = Y_{i+1}}$$

for integral P_{i+1}, Y_{i+1}

(II) Of the solutions of the above, choose $P_{i+1} > 0$, such that

$|(P_{i+1}^{\ 2} - D)|$ has the least value

Then set

$$K_{i+1} = (P_{i+1}^{\ 2} - D)/\,K_i$$

$$Y_{i+1} = (Y_i\,P_{i+1} + X_i)/\,|K_i|$$

$$X_{i+1} = (X_i\,P_{i+1} + DY_i)/\,|K_i|$$

These satisfy $X_{i+1}^{\ 2} - D\,Y_{i+1}^{\ 2} = K_{i+1}$

Iterate the process till $K_{i+1} = \pm 1, \pm 2$ or ± 4, and then solve the equation using *bhāvanā* if necessary.

THE CAKRAVĀLA METHOD

In 1930, Krishnaswami Ayyangar showed that the *cakravāla* procedure always leads to a solution of the equation $X^2 - D Y^2 = 1$.

He also showed that condition (I) is equivalent to the simpler condition

(I') $P_i + P_{i+1}$ is divisible by K_i

Thus, we shall now use the *cakravāla* algorithm in the following form:

To solve $\mathbf{X^2 - D Y^2 = 1}$**:**

Set $X_0 = 1$, $Y_0 = 0$, $K_0 = 1$ and $P_0 = 0$.

Given X_i , Y_i , K_i such that $X_i^2 - D Y_i^2 = K_i$

First find $P_{i+1} > 0$ so as to satisfy:

(I') $P_i + P_{i+1}$ is divisible by K_i

(II) $| P_{i+1}^2 - D |$ is minimum.

THE CAKRAVĀLA METHOD

Then set

$K_{i+1} = (P_{i+1}{}^2 - D)/K_i$

$Y_{i+1} = (Y_i P_{i+1} + X_i)/ |K_i| = a_i Y_i + \varepsilon_i Y_{i-1}$

$X_{i+1} = (X_i P_{i+1} + DY_i)/ |K_i| = P_{i+1} Y_{i+1}$ - sign (K_i) $K_{i+1} Y_i = a_i X_i + \varepsilon_i X_{i-1}$

These satisfy $X_{i+1}{}^2 - D\,Y_{i+1}{}^2 = K_{i+1}$

Iterate till $K_{i+1} = \pm 1, \pm 2$ or ± 4, and then use *bhāvanā* if necessary.

Note: We also need $\boldsymbol{a_i = (P_i + P_{i+1}) / |K_i|}$ **and** $\boldsymbol{\varepsilon_i = (D - P_i^2) / |D - P_i^2|}$ with $\boldsymbol{\varepsilon_0} = 1$.

का सप्तषष्टिगुणिता कृतिरेकयुक्ता का चैकषष्टिनिहता च सखे सरूपा।
स्यान्मूलदा यदि कृतिप्रकृतिर्नितान्तं त्वच्चेतसि प्रवद तात तता लतावत्॥

द्वितीयोदाहरणे न्यासः—

प्र ६१ क १ ज्ये ८ क्षे ३ ।

कुट्टार्थं न्यासः भा १ हा ३ क्षे ८ ।

हरतष्टे धनक्षेपे इति लब्धिगुणौ $\begin{matrix}३\\१\end{matrix}$ । इष्टाहतेति द्वाभ्यामुत्थाप्य जातौ लब्धि गुणौ $\begin{matrix}५\\७\end{matrix}$ । गुणवर्गे ४९ । प्रकृतेः शोधिते १२˙ व्यस्त इति ऋणम् १२˙ इदं क्षेपहृतं जातः क्षेपः ४˙ । अतः प्राग्वज्जाते चतुः क्षेपमूले क ५ ज्ये ३९ ।

इष्टवर्गहृतः क्षेप स्यादित्युपपन्नरूपशुद्धिमूलयोर्भावनार्थं न्यासः—

क $\frac{५}{२}$ ज्ये $\frac{३९}{२}$ क्षे १˙

क $\frac{५}{२}$ ज्ये $\frac{३९}{२}$ क्षे १˙

BHĀSKARA'S EXAMPLE: $X^2 - 61\ Y^2 = 1$

I	P_i	K_i	a_i	ε_i	X_i	Y_i
0	0	1	8	1	1	0
1	8	3	5	-1	8	1
2	7	-4	4	1	39	5
3	9	-5	3	-1	164	21

To find P_1: 0+7, 0+8, 0+9 ... divisible by 1. 8^2 closest to 61. $P_1 = 8$, $K_1 = 3$

To find P_2: 8+4, 8+7, 8+10 ... divisible by 3. 7^2 closest to 61. $P_2 = 7$, $K_2 = -4$

After the second step, we have: $39^2 - 61.5^2 = -4$

Now, since have reached K=-4, we can use *bhāvanā* principle to obtain

$X = (39^2 + 2)\ [(½)\ (39^2 + 1)\ (39^2 + 3) - 1] =$ **1,766,319,049**

$Y = (½)\ (39.5)\ (39^2 + 1)\ (39^2 + 3) =$ **226,153,980**

$1766319049^2 - 61.\ 226153980^2 = 1$

BHĀSKARA'S EXAMPLE: $X^2 - 61\,Y^2 = 1$

I	$\mathbf{P_i}$	$\mathbf{K_i}$	$\mathbf{a_i}$	$\boldsymbol{\varepsilon_i}$	$\mathbf{X_i}$	$\mathbf{Y_i}$
0	0	1	8	1	1	0
1	8	3	5	-1	8	1
2	7	-4	4	1	39	5
3	9	-5	3	-1	164	21
4	6	5	3	1	453	58
5	9	4	4	-1	1,523	195
6	7	-3	5	1	5,639	722
7	8	-1	16	-1	29,718	3,805
8	8	-3	5	-1	469,849	60,158
9	7	4	4	1	2,319,527	296,985
10	9	5	3	-1	9,747,957	1,248,098
11	6	-5	3	1	26,924,344	3,447,309
12	9	-4	4	-1	90,520,989	11,590,025
13	7	3	5	1	335,159,612	42,912,791
14	8	1	16	-1	**1,766,319,049**	**226,153,980**

BHASKARA'S EXAMPLE: $X^2 - 67\ Y^2 = 1$

I	$\mathbf{P_i}$	$\mathbf{K_i}$	$\mathbf{a_i}$	$\boldsymbol{\varepsilon_i}$	$\mathbf{X_i}$	$\mathbf{Y_i}$
0	0	1	8	1	1	0
1	8	-3	5	1	8	1
2	7	6	2	1	41	5
3	5	-7	2	1	90	11
4	9	-2	9	-1	221	27
5	9	-7	2	-1	1,899	232
6	5	6	2	1	3,577	437
7	7	-3	5	1	9,053	1,106
8	8	1	16	1	**48,842**	**5,967**

To find P_1: 0+7, 0+8, 0+9 ... divisible by 1. 8^2 closest to 67. $P_1 = 8$, $K_1 = -3$

To find P_2: 8+4, 8+7, 8+10...divisible by 3. 7^2 closest to 67. $P_2 = 7$, $K_2 = 6$

To find P_3: 7+5, 7+11, 7+17...divisible by 6. 5^2 closest to 67 $P_3 = 5$, $K_3 =-7$

To find P_4: 5+2, 5+9, 5+16...divisible by 7. 9^2 closest to 67 $P_4 = 9$, $K_4 =-2$

Now, since have reached K = -2, we can do *bhāvanā* to find the solution:

$48842^2 - 2.\ 5967^2 = 1$ [$48842 = (221^2 + 67.27^2)/2$ and $5967 = 221.27$]

MODERN SCHOLARSHIP ON CAKRAVĀLA

It is not known whether Bachet, Fermat or their successors in 17th and 18th century were aware of the Indian work on indeterminate equations.

The *Bījagaṇita* of Bhāskara, was translated in to English from the Persian Translation of Ata Allah Rushdi (1634) by Edward Strachey with Notes by Samuel Davis (London, 1813). In 1817, Henry Thomas Colbrooke published *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhascara* (London, 1817), which included a tranlsation of *Gaṇitādhyāya* and *Kuṭṭakādhyāya* of *Brāhmasphuṭa-siddhānta*, and the *Līlāvatī* and *Bījagaṇita* of Bhāskara.

The true nature of *cakravāla* method was not understood for long. In the second edition (1910) of his work on Diophantus, Edward Heath notes:

> On the Indian method Hankel [1874] says, 'It is above all praise; it is certainly the finest thing which was achieved in theory of numbers before Lagrange'; and although this may seem an exaggeration when we think of the extraordinary achievements of Fermat, it is true that the Indian method is, remarkably though, the same as that which was rediscovered and expounded by Lagrange in 1768.

ANALYSIS OF CAKRAVĀLA PROCESS

In 1930, A. A. Krishnaswami Ayyangar (1892-1953) presented a detailed analysis of the *cakravāla* process. He explained how it is different from and more optimal than the Euler-Lagrange process based on the simple continued fraction expansion of √D. He also showed that the *cakravāla* process always leads to a solution of the *vargaprakṛti* equation with K=1.

Let us consider the equations

$$X_i^{\,2} - D\, Y_i^{\,2} = K_i$$

$$P_{i+1}^{\,2} - D.1^2 = P_{i+1}^{\,2} - D$$

By doing *bhāvanā* of these, and dividing by K_i^2, we get

$$\mathbf{[(X_i\, P_{i+1} + DY_i)/\,|K_i|]^2 - D\,[(Y_i\, P_{i+1} + X_i)/\,|K_i|]^2 = (P_{i+1}^{\,2} - D)\,/\,K_i}$$

If we assume that X_i, Y_i and K_i are mutually prime, and if we choose P_{i+1} such that $Y_{i+1} = [(Y_i\; P_{i+1} + X_i)/\,|K_i|]$ is an integer, then it can be shown that X_{i+1}=$[(X_i\; P_{i+1} + DY_i)/K_i|]$ and K_{i+1}= $(P_{i+1}^{\,2} - D)\,/K_i$ are both integers.

Further, we have

$$X_{i+2} = [(X_{i+1}P_{i+2} + DY_{i+1})/ |K_{i+1}|]$$

$$= X_{i+1} [(P_{i+2} + P_{i+1})/ |K_{i+1}|] + X_i (D - P_{i+1}^2)/ |K_i| |K_{i+1}|$$

$$= a_{i+1} X_{i+1} + \varepsilon_{i+1} X_i$$

and similarly for Y_{i+2}.

Therefore, instead of using the *kuṭṭaka* process for finding P_{i+2}, we can use the condition that

$P_{i+1} + P_{i+2}$ is divisible by K_{i+1}.

ANALYSIS OF CAKRAVĀLA PROCESS

Krishnaswami Ayyangar, then proceeds to an analysis of the quadratic forms (K_i, P_{i+1}, K_{i+1}) which satisfy $P_{i+1}^2 - K_i K_{i+1} = D$.

Notation: (A, B, C) stands for the quadratic form $Ax^2+2Bxy+Cy^2$

The form $(K_{i+1}, P_{i+2}, K_{i+2})$, which is obtained from (K_i, P_{i+1}, K_{i+1}) by the *cakravāla* process, is called the successor of the latter.

Ayyangar defines a quadratic form (A, B, C) to be a Bhāskara form if

$$\mathbf{A^2 + (C^2/4) < D \text{ and } C^2 + (A^2/4) < D}$$

He shows that the successor of a Bhāskara form is also a Bhāskara form and that two different Bhāskara forms cannot have the same successor.

ANALYSIS OF CAKRAVĀLA PROCESS

Krishnaswami Ayyangar considers the general case when we start the *cakravāla* process with an arbitrary initial solution

$$X_0^2 - D\, Y_0^2 = K_0$$

He shows that if $|K_0| > \sqrt{D}$, then the absolute values of the successive K_i decrease monotonically, till say K_m, after which we have

$$|\mathbf{K_i}| < \sqrt{\mathbf{D}} \text{ for } i > m.$$

He also shows that

$$|\mathbf{P_i}| < \mathbf{2}\sqrt{\mathbf{D}} \text{ for } i > m.$$

Since $|K_i|$ cannot go on decreasing, for some $r>m$ we have $|K_{r+1}| > |K_r|$. It can then be shown that (K_r, P_{r+1}, K_{r+1}) and all the succeeding forms will be Bhāskara forms.

ANALYSIS OF CAKRAVĀLA PROCESS

If we start with the initial solution $X_0 = 1$, $Y_0 = 0$ and $K_0 = 1$, then we see that *cakravāla* process leads to $P_1 = X_1 = d$, where $d > o$ is the integer such that d^2 is the square nearest to D. Also $Y_0 = 1$ and $K_1 = d^2 - D$.

Ayyangar shows that $(K_0, P_1, K_1) \equiv (1, d, d^2 - D)$ is a Bhāskara form. So is the form $(d^2 - D, d, 1)$ which is equivalent to it.

Since the values of K_i, P_i are bounded, the Bhāskara forms will have to repeat in a cycle and the first member of the cycle is the same as the first Bhāskara form which is obtained in the course of *cakravāla*.

Finally, Ayyangar shows that two different cycles of Bhāskara forms are non-equivalent, and that all equivalent Bhāskara forms belong to the same cycle.

ANALYSIS OF CAKRAVĀLA PROCESS

Ayyangar sets up an association between a Bhāskara form (K_i, P_{i+1}, K_{i+1}) an equivalent Gauss form $(K_i', P_{i+1}', K_{i+1}')$, which satisfies

$$\sqrt{D} - P_{i+1}' < |K_i'| < \sqrt{D} + P_{i+1}'.$$

If $P_{i+1} < \sqrt{D}$, then $(K_i', P_{i+1}', K_{i+1}') \equiv (K_i, P_{i+1}, K_{i+1})$

If $P_{i+1} > \sqrt{D}$, then $K_i' = K_i$, $P_{i+1}' = P_{i+1} - |K_i|$ and $K_{i+1}' = 2P_{i+1} - |K_i| - |K_{i+1}|$

In this way a Bhāskara cycle can be converted to a unique Gauss cycle and vice versa, from which the above results follow.

Thus, whatever initial solution we may start with, the *cakravāla* process takes us to a cycle of equivalent Bhāskara forms and since the Bhāskara form $(d^2-D, d, 1)$ is in this equivalence class, the *cakravāla* process leads to a solution corresponding to $K = 1$.

FERMAT'S CHALLENGE TO BRITISH MATHEMATICIANS (1657)

In February 1657, Pierre de Fermat (1601-1665) wrote to Bernard Frenicle de Bessy asking him for a general rule "for finding, when any number not a square is given, squares which, when they are respectively multiplied by the given number and unity added to the product, give squares." If Frenicle is unable to give a general solution, Fermat said, can he at least give the smallest values of x and y which will satisfy the equations $\mathbf{61x^2 + 1 = y^2}$ and $109x^2 + 1 = y^2$.

At the same time Fermat issued a general challenge, addressed to the mathematicians in northern France, Belgium and England, where he says:

> "There is hardly anyone who propounds purely arithmetical questions, hardly anyone who understands them. Is this due to the fact that up to now arithmetic has been treated geometrically rather than arithmetically? This has indeed generally been the case both in ancient and modern works; even Diophantus is an instance. For, although he was freed himself from geometry a little more than others in that he confines his analysis to the consideration of rational numbers, yet even there geometry is not absent…

FERMAT'S CHALLENGE TO BRITISH MATHEMATICIANS (1657)

"Now, arithmetic has so to speak, a special domain of its own, the theory of integral numbers. This was only lightly touched upon by Euclid in his *Elements,* and was not sufficiently studied by those who followed him…

To arithmeticians, therefore by way of lighting up the road to be followed, I propose the following theorem to be proved or problem to be solved. If they succeed in discovering the proof or solution, they will admit that questions of this kind are not inferior to the more celebrated questions in geometry in respect of beauty, difficulty or method of proof.

Given any number whatever which is not a square, there are also given infinite number of squares such that, if the square is multiplied into the given number and unity is added to the product, the result is a square....

Eg. Let it be required to find a square such that, if the product of the square and the number 149, or 109, or 433 etc. be increased by 1, the result is a square."

BROUNKER-WALLIS SOLUTION

Fermat's Challenge was addressed to William Brouncker (1620-1684) and John Wallis (1616-1703). Brouncker's first response merely contained rational solutions and this led to Fermat complaining (in a letter to the interlocutor Kenelm Digby in August 1657) that they were no solutions at all to the problem that he had posed.

Brouncker then worked out his method of integral solutions which he sent to Wallis to be communicated to Fermat. Wallis describes the method of solution in two letters dated December 17, 1657 and January 30, 1658. Later in 1658, Wallis published the entire correspondence as *Commercium Epistolicum*. He also outlined the method in his *Algebra* published in English in 1685 and in Latin in 1693.

We do not know what method Fermat had for the solution of the problem he posed. Of course he communicated to the English mathematicians that he "willingly and joyfully acknowledges" the validity of their solutions. He however wrote to Huygens in 1659 that the English had failed to give "a general proof", which according to him could only be obtained by the "method of descent".

EULER-LAGRANGE METHOD OF SOLUTION

In a letter to Goldbach written on August 10, 1730, Leonhard Euler (1707-1787) mentions the equation $X^2 - 8\ Y^2 = 1$ as a special case of "Pell's Equation". He notes that "such problems have been agitated between Wallis and Fermat... and the Englishman Pell devised for them a peculiar method described in Wallis's works."

Citing the above André Weil notes:

> "Pell's name occurs frequently in Wallis's *Algebra*, but never in connection with the equation $X^2 – N\ Y^2 = 1$ to which his name, because of Euler's mistaken attribution, has remained attached; since its traditional designation as 'Pell's equation' is unambiguous and convenient, we will go on using it even though it is historically wrong."

In a paper "*De solution problematum Diophanterum per numerous integros*" written in 1730, Euler describes Wallis method. He also shows that from one solution of "Pell's equation" an infinite number of solutions can be found and also remarks that they give good approximations to square-roots.

EULER-LAGRANGE METHOD OF SOLUTION

In his letters to Goldbach in 1753 and 1755 Euler speaks of certain improvements he had made in the "Pellian method".

In a paper, read in 1759 but published in 1765 (1767), entitled "*De Usu novi algoritmi in problemate Pelliano solvendo*" Euler describes the method of solving $X^2 - D\,Y^2 = 1$ by the simple continued fraction expansion of √D. He gives a table of cycles (of partial quotients) for all non-square integers from 2 to 120 and also notes their various properties.

In a paper which was published earlier in 1762-3 (1764) Euler proves the *bhāvanā* principle and called it "*Theorema Elegantissimum*".

Euler also wrote a paper in 1773 (published in 1783) on "New Aids" for solving the Pell's equation, where he describes how the equation can be solved if solution is known for K = -1, 2, -2, 4.

By then, in a set of three papers presented to the Berlin Academy in 1768, 1769 and 1770, Joseph Louis Lagrange (1736-1813) had already worked out the complete theory of continued fractions and their applications to Pell's equation along with all the necessary proofs.

RELATION WITH CONTINUED FRACTION EXPANSION

A simple continued fraction is of the form (a_i are positive integers for i>0)

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

This is denoted by [a_0, a_1, a_2, a_3, ...] or by

$$a_0 + \frac{1}{a_1 +}\frac{1}{a_2 +}\cdots$$

Given any real number α, to get the continued fraction expansion, take

$a_0 = [\alpha]$ the integral part of α.

Let $\alpha_1 = 1/(\alpha - [\alpha])$. Then we take $a_1 = [\alpha_1]$

Let $\alpha_2 = 1/(\alpha_1 - [\alpha_1])$. Then we take $a_2 = [\alpha_2]$, and so on.

a_0, a_1, a_2, .. are called partial quotients; α_1, α_2, ... are the complete quotients.

RELATION WITH CONTINUED FRACTION EXPANSION

The k-th convergent of the continued fraction $[a_0, a_1, a_2, a_3, \ldots]$ is given by

$$A_k/B_k = [a_0, a_1, a_2, a_3, \ldots, a_k]$$

A_k, B_k satisfy the recurrence relations:

$$A_0 = a_0,\ A_1 = a_1 a_0 + 1,$$

$$A_k = a_k A_{k-1} + A_{k-2} \text{ for } k \geq 2$$

$$B_0 = 1,\ B_1 = a_1,$$

$$B_k = a_k B_{k-1} + B_{k-2} \text{ for } k \geq 2$$

The convergents also satisfy

$$A_j B_{j-1} - A_{j-1} B_j = (-1)^{j-1}$$

RELATION WITH CONTINUED FRACTION EXPANSION

Example

$149/17 = [8, 1, 3, 4]$

The convergents are

$A_0/B_0 = 8/1$, $A_1/B_1 = 9/1$, $A_2/B_2 = 35/4$, $A_3/B_3 = 149/17$

We have

$A_3 B_2 - A_2 B_3 = 149.4 - 35.17 = 1$

This is very similar to the *kuṭṭaka* method for solving $149x - 17y = 1$.

Note: The simple continued fraction expansion of a real number does not terminate if the number is irrational. For instance

$(1+\sqrt{5})/2 = [1, 1, 1, 1, ...]$,

$e = [2, 1, 2, 1, 1, 4, 1,1, 6, 1, 1, 8, 1, 1, ...]$.

RELATION WITH CONTINUED FRACTION EXPANSION

It was noted by Euler that the simple continued fraction of √D is always periodic and is of the form

$$\sqrt{D} = [a_0, \overline{a_1, \ldots, a_{h-1}, a_{h-1}, \ldots, a_1, 2a_0}] \text{ if } k = 2h-1,$$
$$\sqrt{D} = [a_0, \overline{a_1, \ldots, a_{h-1}, a_h, a_{h-1}, \ldots, a_1, 2a_0}] \text{ if } k = 2h,$$

where *k* is the length of the period, and the convergents A_{k-1}, B_{k-1} satisfy

$$A_{k-1}^2 - DB_{k-1}^2 = (-1)^k$$

Further, all the solutions of, $X^2 - D Y^2 = 1$ can be obtained by composing (*bhāvanā*) of the above solution with itself.

These results were later proved by Lagrange.

Example: To solve $X^2 - 13Y^2 = 1$

$$\sqrt{13} = 3 + \overline{\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{6+}}$$

$A_4/B_4 = 18/5$ and we have $18^2 - 13.5^2 = -1$, leading to $649^2 - 13.180^2 = 1$

BHĀSKARA SEMI-REGULAR CONTINUED FRACTIONS

In a couple of papers which appeared during 1938-41, Krishnaswami Ayyangar showed that the *cakravāla* procedure actually corresponds to a "semi-regular continued fraction" expansion of √D.

A semi-regular continued fraction is of the form

$$a_0 + \boldsymbol{\varepsilon_1} / a_{1+} \ \ \boldsymbol{\varepsilon_2} / a_{2+} \ \ \boldsymbol{\varepsilon_3} / a_{3+} \ \ldots$$

where $\varepsilon_i = \pm 1$, $a_i \geq 1$ for $i \geq 1$, and $a_i + \varepsilon_{i+1} \geq 1$ for $i \geq 1$.

Then the convergents satisfy the relations

$$A_0 = a_0,\ A_1 = a_1 a_0 + \varepsilon_1,$$

$$A_j = a_j A_{j-1} + \varepsilon_j A_{j-2} \text{ for } j \geq 2$$

$$B_0 = 1,\ B_1 = a_1,$$

$$B_j = a_j B_{j-1} + \varepsilon_j B_{j-2} \text{ for } j \geq 2$$

BHĀSKARA SEMI-REGULAR CONTINUED FRACTIONS

Krishnaswami Ayyangar showed that the *cakravāla* method of Bhāskara corresponds to a periodic semi-regular continued function (Nearest Square Continued Fraction) expansion

$$\sqrt{\mathbf{D}} = a_0 + \varepsilon_1 / a_{1+} \;\; \varepsilon_2 / a_{2+} \;\; \varepsilon_3 / a_{3+} \ldots$$

where $a_i = (\mathbf{P_i} + \mathbf{P_{i+1}}) / |\mathbf{K_i}|$, $\varepsilon_i = (\mathbf{D} - \mathbf{P_i}^2) / |\mathbf{D} - \mathbf{P_i}^2|$ and the convergents are related to the solutions $\mathbf{A_j} = \mathbf{X_{j+1}}$ **and** $\mathbf{B_j} = \mathbf{Y_{j+1}}$.

Note: The Simple Continued Fraction of Euler-Lagrange and the Nearest Integer Continued Fraction can also be generated by a *cakravāla* type of algorithm if we replace the condition II respectively by

(II') $\mathbf{D} - \mathbf{P_{i+1}}^2 > 0$ **and is minimum**

(II'') $|\mathbf{P_{i+1}} - \sqrt{\mathbf{D}}|$ **is minimum**

[The Nearest Integer Continued Fraction expansion is implicit in the variant of the *cakravāla* method discussed by Nārāyaṇa Paṇḍita in his *Gaṇitakaumudī* (c.1356). It was discovered as a more optimal method for solving "Pell’s equation" than SCF by B. Minnigerode in 1873]

EULER-LAGRANGE METHOD FOR $X^2 - 67\ Y^2 = 1$

I	P_i	K_i	a_i	ε_i	X_i	Y_i
0	0	1	8	1	1	0
1	8	-3	5	1	8	1
2	7	6	2	1	41	5
3	5	-7	1	1	90	11
4	**2**	**9**	**1**	**1**	**131**	**16**
5	7	-2	7	1	221	27
6	**7**	**9**	**1**	**1**	**1678**	**205**
7	2	-7	1	1	1899	232
8	5	6	2	1	3577	437
9	7	-3	5	1	9053	1106
10	8	1	16	1	48842	5967

The steps which are skipped in cakravāla are highlighted

The corresponding simple continued fraction expansion is

$$\sqrt{67} = 8 + \overline{\frac{1}{5+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{7+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{5+}\ \frac{1}{16+}}$$

The Bhāskara nearest square continued fraction is given by

$$\sqrt{67} = 8 + \overline{\frac{1}{5+}\ \frac{1}{2+}\ \frac{1}{2+}\ \frac{-1}{9+}\ \frac{-1}{2+}\ \frac{1}{2+}\ \frac{1}{5+}\ \frac{1}{16+}}$$

EULER-LAGRANGE METHOD FOR $X^2 - 61 Y^2 = 1$

I	P_i	K_i	a_i	ε_i	X_i	Y_i
0	0	1	7	1	1	0
1	7	-12	1	1	**7**	**1**
2	5	3	4	1	8	1
3	7	-4	3	1	39	5
4	5	9	1	1	**125**	**16**
5	4	-5	2	1	164	21
6	6	5	2	1	453	58
7	4	-9	1	1	**1070**	**137**
8	5	4	3	1	1523	195
9	7	-3	4	1	5639	722
10	5	12	1	1	**24079**	**3083**
11	7	-1	14	1	29718	3805

The steps which are skipped in *cakravāla* are highlighted

EULER-LAGRANGE METHOD FOR $X^2 - 61\ Y^2 = 1$ (CONTD)

12	7	12	1	1	**440131**	**56353**
13	5	-3	4	1	469849	60158
14	7	4	3	1	2319527	296985
15	5	-9	1	1	**7428430**	**951113**
16	4	5	2	1	9747967	1248098
17	6	-5	2	1	26924344	3447309
18	4	9	1	1	**63596645**	**8142716**
19	5	-4	3	1	90520989	11590025
20	7	3	4	1	335159612	42912791
21	5	-12	1	1	**1431159437**	**183241189**
22	7	1	14	1	1766319049	226153980

The corresponding simple continued fraction expansion is

$$\sqrt{61} = 7 + \overline{\frac{1}{1+}\ \frac{1}{4+}\ \frac{1}{3+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{3+}\ \frac{1}{4+}\ \frac{1}{1+}\ \frac{1}{14+}}$$

The Bhāskara nearest square continued fraction is given by

$$\sqrt{61} = 8 + \overline{\frac{-1}{5+}\ \frac{1}{4+}\ \frac{-1}{3+}\ \frac{1}{3+}\ \frac{-1}{4+}\ \frac{1}{5+}\ \frac{-1}{16+}}$$

BHĀSKARA OR NEAREST SQUARE CONTINUED FRACTION

In the continued fraction development of $\sqrt{D}$, the complete quotients are quadratic surds which may be expressed in the standard form $(P + \sqrt{D})/Q$, where P, Q and $(D-P^2)/Q$ are integers prime to each other.

If $a = [(P + \sqrt{D})/Q]$ is the integral part of $(P + \sqrt{D})/Q$, then we can have

$$(P + \sqrt{D})/Q = a + Q'/(P' + \sqrt{D}) \qquad \text{(I)}$$

$$(P + \sqrt{D})/Q = (a + 1) - Q''/(P'' + \sqrt{D}) \qquad \text{(II)}$$

where the surds in the rhs are also in the standard form.

In the Bhāskara or Nearest Square Continued Fraction development we choose a as the partial quotient

(i) if $|P'^2-D| < |P''^2-D|$,

ii) or if $|P'^2-D| = |P''^2-D|$ and $Q<0$. Then we set $\varepsilon = 1$.

Otherwise we choose $a + 1$ as the partial quotient and set $\varepsilon = -1$.

Note: If we start with $\sqrt{D}$, we always have $P_i \geq 0$ and $Q_i > 0$ and

$$K_i = (-1)^i \varepsilon_1 \varepsilon_2 \ldots \varepsilon_i Q_i$$

BHĀSKARA OR NEAREST SQUARE CONTINUED FRACTION

Krishnaswami Ayyangar showed that the Bhāskara or nearest square continued fraction of √D is of the form

$$\sqrt{D} = a_0 + \overline{\frac{\varepsilon_1}{a_1+}\;\frac{\varepsilon_2}{a_2+}\;\frac{\varepsilon_3}{a_3+}\;\cdots\;\frac{\varepsilon_{k-1}}{a_{k-1}+}\;\frac{\varepsilon_k}{2a_0+}}$$

where *k* is the period. Further, it has the following symmetry properties:

Type I: There is no complete quotient of the form $[p+q+\sqrt{(p^2+q^2)}]/p$, where $p>2q>0$ are mutually prime integers. In this case, the Bhāskara continued fraction for √D has same symmetry properties as in the case of Simple Continued Fraction expansion.

$$\begin{aligned} a_v &= a_{k-v}, & 1 \le v \le k-1, \\ Q_v &= Q_{k-v}, & 1 \le v \le k-1, \\ \epsilon_v &= \epsilon_{k+1-v}, & 1 \le v \le k, \\ P_v &= P_{k+1-v}, & 1 \le v \le k. \end{aligned}$$

Examples of Type I:

$$\sqrt{61} = 8 + \overline{\frac{-1}{5+}\,\frac{1}{4+}\,\frac{-1}{3+}\,\frac{1}{3+}\,\frac{-1}{4+}\,\frac{1}{5+}\,\frac{-1}{16+}}$$

$$\sqrt{67} = 8 + \overline{\frac{1}{5+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{-1}{9+}\,\frac{-1}{2+}\,\frac{1}{2+}\,\frac{1}{5+}\,\frac{1}{16+}}$$

Type II: There is a complete quotient of the form [p+q+√(p^2+q^2)]/p, where p>2q>0 are mutually prime integers. In such a case, the period *k* ≥4 and is even and there only one such complete quotient which occurs at *k*/2. The symmetry properties are same as for Type I, except that

$$a_{\frac{k}{2}} = 2, \epsilon_{\frac{k}{2}} = -1, \epsilon_{\frac{k}{2}+1} = 1, a_{\frac{k}{2}-1} = a_{\frac{k}{2}+1} + 1, P_{\frac{k}{2}} \neq P_{\frac{k}{2}+1}$$

Examples of Type II:

$$\sqrt{58} = 8 + \overline{\frac{-1}{3+}\,\frac{-1}{2+}\,\frac{1}{2+}\,\frac{-1}{16+}}$$

$$\sqrt{97} = 10 + \overline{\frac{-1}{7+}\,\frac{-1}{3+}\,\frac{-1}{2+}\,\frac{1}{2+}\,\frac{-1}{7+}\,\frac{-1}{20+}}$$

LEHMER ON BHĀSKARA CONTINUED FRACTION

The American number theorist Derrick Henry Lehmer (1905-1991) reviewed the work of Krishnaswami Ayyangar in the Mathematical Reviews (1944):

> "These papers are concerned with a peculiar semi-regular continued fraction algorithm somewhat akin to the nearest integer algorithm. The expansion is defined only for the standard quadratic surd... and is presumably based on Bhāskara's 'cyclic method' of solving the Pell equation, although the author does not trace the connection in detail...
>
> The author proves that the period is either symmetric, as in the regular case, or else 'almost symmetric', that is symmetric except for three central asymmetric partial quotients...
>
> The theory of this continued fraction in most respects closely parallels the classical regular case of Lagrange. The slight blemishes characteristic of the new (or should one say 12th century) algorithm do not seem to be compensated by any useful features other than that possessed by the nearest integer algorithm...."

RECENT WORK ON "PELL'S EQUATION"

"Pell's equation" continues to be an important subject of current research.

The period k(D) of the simple continued fraction expansion of √D is a good measure of the number of steps needed to solve the equation. And this is known to fluctuate wildly between as low a value as 1 and an upper bound which is of the order of (√D)log(D). The solutions are similarly bounded above by a multiple of D exp(√D)

For instance, for D=1620, k(D) = 1 and we have the roots (161, 4)

For D=1621, k(D) = 78 (the NSCF has a period of 56), and the larger root has 76 digits!

While it is true that even writing down the solutions involves exponential time (in terms of input length logD), various algebraic number theoretic techniques are being developed to obtain a significant number of digits of the solution by "faster" means. Recently Hallgren (2002) has come up with a polynomial-time quantum algorithm for the same purpose.

MID-POINT CRITERIA

Recently, Mathews, Robertson and White (2010) have worked out the mid-point criteria for the Bhāskara continued fraction expansion of $\sqrt{D}$.

In the case of the Simple Continued Fraction expansion of $\sqrt{D}$, the mid-point criteria were given by Euler:

If $Q_{h-1} = Q_h$ (or $|K_{h-1}| = |K_h|$), then the period $k = 2h-1$ and

$$A_{k-1} = A_{h-1}B_{h-1} + A_{h-2}B_{h-2}$$

$$B_{k-1} = B_{h-1}^2 + B_{h-2}^2$$

which satisfy $A_{k-1}^2 - DB_{k-1}^2 = -1$

If $P_h = P_{h+1}$, then the period $k = 2h$ and

$$A_{k-1} = A_{h-1}B_h + A_{h-2}B_{h-1}$$

$$B_{k-1} = B_{h-1}(B_h + B_{h-2})$$

which satisfy $A_{k-1}^2 - DB_{k-1}^2 = 1$

MID-POINT CRITERIA

Mathews et al have obtained the following mid-point criteria for the Bhāskara or the Nearest Square Continued Fraction expansion of √D:

If $Q_{h-1} = Q_h$ (or $|K_{h-1}| = |K_h|$), then the period $k = 2h-1$ and

$$A_{k-1} = A_{h-1}B_{h-1} + \varepsilon_h A_{h-2}B_{h-2}$$

$$B_{k-1} = B_{h-1}^2 + \varepsilon_h B_{h-2}^2$$

which satisfy $A_{k-1}^2 - DB_{k-1}^2 = -\varepsilon_h$

If $P_h = P_{h+1}$, then the period $k = 2h$ and

$$A_{k-1} = A_{h-1}B_h + \varepsilon_h A_{h-2}B_{h-1}$$

$$B_{k-1} = B_{h-1}(B_h + \varepsilon_h B_{h-2})$$

which satisfy $A_{k-1}^2 - DB_{k-1}^2 = 1$

In the Type I case, the mid-point will in variably satisfy one of the above two criteria.

MID-POINT CRITERIA

In the Type II case, the following is the mid-point criterion:

When $Q_h = |K_h|$ is even and $P_h = Q_h + (½)Q_{h-1} = |K_h| + (½)|K_{h-1}|$ and $\varepsilon_h = 1$, then $k = 2h$ and

$$A_{k-1} = A_h B_{h-1} - B_{h-2}(A_{h-1} - A_{h-2})$$

$$B_{k-1} = 2B_{h-1}^2 - B_h B_{h-2}$$

which satisfy $A_{k-1}^2 - DB_{k-1}^2 = 1$

These mid-point criteria serve to further simplify the computation of the solution.

Note: The mid-point criteria for the Nearest Integer Continued Fraction expansion, obtained by Williams and Buhr (1979), are considerably more complicated than the above mid-point criteria for NSCF.

We have already remarked that the *cakravāla* process skips certain steps in the Euler-Lagrange process. Sometimes the period of Euler-Lagrange continued fraction expansion could be double (or almost double) the period of Bhāskara expansion. For instance, for D=13, 44, 58, we have the following:

$$\text{BCF: } \sqrt{13} = 4 + \overline{\frac{-1}{2+}\ \frac{1}{2+}\ \frac{-1}{8+}}$$

$$\text{SCF: } \sqrt{13} = 3 + \overline{\frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{6+}}$$

$$\text{BCF: } \sqrt{44} = 7 + \overline{\frac{-1}{3+}\ \frac{-1}{4+}\ \frac{-1}{3+}\ \frac{-1}{14+}}$$

$$\text{SCF: } \sqrt{44} = 6 + \overline{\frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{12+}}$$

$$\text{BCF: } \sqrt{58} = 8 + \overline{\frac{-1}{3+}\ \frac{-1}{2+}\ \frac{1}{2+}\ \frac{-1}{16+}}$$

$$\text{SCF: } \sqrt{58} = 7 + \overline{\frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{14+}}$$

OPTIMALITY OF CAKRAVĀLA METHOD

We may note that whenever there is a 'unisequence' (1,1,...,1) of partial quotients of length n, the *cakravāla* process skips exactly ($n/2$) steps if n is even, and $(n+1)/2$ steps if n is odd.

In a series of papers (1960-63), Selenius has shown that the *cakravāla* process is 'ideal' in the sense that, whenever there is such a 'unisequence', only those convergents A_i/B_i are retained for which $B_i \left| A_i - B_i \sqrt{D} \right|$ are minimal.

Recently, Mathews *et al* (2010) have shown that the period of Bhāskara or Nearest Square Continued Fraction is the same as that of the Nearest Integer Continued Fraction. They also estimate that the ratio of this period to that of simple continued fraction is $\log_2 [(1+\sqrt{5})/2] \approx 0.6942419136...$

OPTIMALITY OF CAKRAVĀLA METHOD

n	$\Pi\ (n)$	$P\ (n)$	$\Pi\ (n)/P\ (n)$
1,000,000	152,198,657	219,245,100	0.6941941
2,000,000	417,839,927	601,858,071	06942499
3,000,000	755,029,499	1,087,529,823	0.6942609
4,000,000	1,149,044,240	1,655,081,352	0.6942524
5,000,000	1,592,110,649	2,293,328,944	0.6942356
6,000,000	2,078,609,220	2,994,112,273	0.6942322
7,000,000	2,604,125,007	3,751,067,951	0.6942356
8,000,000	3,165,696,279	4,559,939,520	0.6944208
9,000,000	3,760,639,205	5,416,886,128	0.6942437
10,000,000	4,387,213,325	6,319,390,242	0.6942463

$\Pi(n)$ is the sum of the NSCF period lengths of $\sqrt{D}$ up to n, D not a square, and $P(n)$ is the same for RCF.

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