On the spectrum of the Laplacian

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$$u = 0 \quad \text{on } \partial \Omega$$

where

$$\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$$

and $\partial \Omega$ denotes the boundary of $\Omega$. 
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$N = 1$: vibrating string which is fixed at both ends.

$N = 2$: vibrating membrane (drum) fixed along the boundary.
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We get that the only non-trivial solutions are

$$\lambda_n = n^2 \pi^2, \quad u_n = C \sin n\pi x, \quad n \in \mathbb{N}$$

where $C$ is any real constant. If we fix $C = \sqrt{2}$, we get

$$\int_0^1 u_n^2(x) \, dx = 1, \text{ for all } n \in \mathbb{N}.$$
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0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty
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- Notice that

\[
\int_0^1 u_n(x) u_m(x) \, dx = \delta_{nm}
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and, from the theory of Fourier series, we know that \( \{u_n\} \) forms an orthonormal basis of \( L^2(0, 1) \) (Fourier sine series).
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- \( u_1 > 0 \) in \((0,1)\).

- \( u_n \) has exactly \( n - 1 \) zeros in \((0,1)\).
Let $\Omega \subset \mathbb{R}^N$ be a reasonably smooth domain. Then $H^1(\Omega)$ is the completion of $C^\infty(\Omega)$ with respect to the norm

$$
\|u\|_{1,\Omega} = \left[ \int_\Omega (|\nabla u(x)|^2 + |u(x)|^2) \, dx \right]^{\frac{1}{2}}
$$

where $x = (x_1, x_2, \cdots, x_N)$ and $dx = dx_1 dx_2 \cdots dx_N$. 

Poincaré’s Inequality states that the semi-norm $|u|_{1,\Omega} = \left[ \int_\Omega |\nabla u(x)|^2 \, dx \right]^{\frac{1}{2}}$ is also a norm for $H^1_0(\Omega)$, which is equivalent to the norm $\|u\|_{1,\Omega}$. 

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is also a norm for $H^1_0(\Omega)$, which is equivalent to the norm $\| u \|_{1,\Omega}$.
Equivalently, $H^1(\Omega)$ can be thought of as the space of $L^2(\Omega)$ ‘functions’ whose distributional derivatives of the first order are also in $L^2(\Omega)$ and $H^1_0(\Omega)$ is the closed subspace of ‘functions’ in $H^1(\Omega)$ which ‘vanish’ on the boundary $\partial \Omega$. 
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These are Hilbert spaces with the following inner-products:

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Its weak form is to find $v \in H^1_0(\Omega)$ such that

$$
\int_{\Omega} \nabla v \cdot \nabla w \, dx = \int_{\Omega} fw
$$

for all $w \in H^1_0(\Omega)$. This problem has a unique solution and we define $G : L^2(\Omega) \to H^1_0(\Omega)$ by $G(f) = v$. This is a continuous operator and if we compose it with the inclusion $H^1_0(\Omega) \subset L^2(\Omega)$, which is compact (Rellich’s theorem) we get that $G$ is a compact operator of $L^2(\Omega)$ into itself and it is easy to see that it is self-adjoint as well.
If \((u, \lambda)\) solves the original eigenvalue problem, then, in the new notation we have

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u = G(\lambda u)
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and, since \(\lambda > 0\), we can write

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Thus, from the spectral theory of compact self-adjoint operators on a Hilbert space, we deduce that there exists a sequence \(\{\lambda_n\}\) of positive eigenvalues increasing to infinity and an associated orthonormal family of eigenfunctions \(\{u_n\}\) which forms an orthonormal basis for \(L^2(\Omega)\).
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Let us write

\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty \]

with the \(\lambda_n\) being repeated as many times as the dimension of the corresponding eigenspace.
Example

\[ \Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2. \] Then, it is easy to see that \( \lambda_{nm} = \pi^2 (n^2 + m^2) \) is an eigenvalue with corresponding eigenfunction

\[ u_{nm} = 2 \sin n\pi x \sin m\pi y. \]

That these are the only ones needs proof and follows from the fact that \( \{u_{nm}\} \) is a complete orthonormal basis for \( L^2(\Omega) \). Thus, \( \lambda_1 = 2\pi^2 \) while \( \lambda_2 = \lambda_3 = 5\pi^2 \) corresponding to \( n = 1, m = 2 \) and \( n = 2, m = 1 \) and the space of eigenfunctions is two dimensional spanned by \( 2 \sin \pi x \sin 2\pi y \) and \( 2 \sin 2\pi x \sin \pi y \).
Example

\( \Omega \) is the unit disc in \( \mathbb{R}^2 \). In polar coordinates, we have

\[
- \left[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} \right] = \lambda u.
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$$- \left[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right] = \lambda u.$$ 

We look for solutions of the form $u(r, \theta) = v(r)w(\theta)$ and this leads us to look at

$$w'' + kw = 0, \ w \text{ is } 2\pi - \text{periodic}$$

and

$$v'' + \frac{1}{r} v' + \left( \lambda - \frac{k}{r^2} \right) = 0$$

with $v'(0) = v(1) = 0$. The first equation implies that $k = n^2$, $n \in \{0\} \cup \mathbb{N}$ and substituting it in the second leads us to the Bessel’s equation. In particular, $u_1$ corresponds to $k = 0$ and is a radial function and $\lambda_1$ comes from the first zero of the Bessel function $J_0$:

$$\lambda_1 = j^2_{0,1}, \ u_1 = CJ_0(j_{0,1}r).$$
If \( j_{0,l} \) is the \( l \)-th zero of \( J_0 \), then \( j_{0,l}^2 \) is a simple eigenvalue with eigenfunction \( C J_0(j_{0,l},r) \) which is also radial. While \( u_1 \) is positive in \( \Omega \), the others change sign.
If $j_{0,l}$ is the $l$-th zero of $J_0$, then $j_{0,l}^2$ is a simple eigenvalue with eigenfunction $CJ_0(j_{0,l}r)$ which is also radial. While $u_1$ is positive in $\Omega$, the others change sign.

If we take $k = n^2$, $n \in \mathbb{N}$, then for $n, l \geq 1$ we have the double eigenvalue $j_{n,l}^2$ with eigenspace spanned by

$$CJ_n(j_{n,l}r) \cos n\theta, \text{ and } CJ_n(j_{n,l}r) \sin n\theta.$$
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Given an orthonormal basis of eigenfunctions \( \{u_n\} \) corresponding to the eigenvalues \( \{\lambda_n\} \) listed in increasing order, taking into account the multiplicity, set, for \( k \in \mathbb{N} \),

\[ V_k = \text{span}\{u_1, \ldots, u_k\} \]
Variational Characterization

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Then, for \(k \in \mathbb{N}\),

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\lambda_k = R(u_k) = \max_{v \in V_k, v \neq 0} R(v) = \min_{v \perp V_{k-1}, v \neq 0} R(v) = \min_{V \subset H_0^1(\Omega), \dim V = k} \max_{v \in V, v \neq 0} R(v)
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In particular,

\[ \lambda_1 = \min_{v \in H^1_0(\Omega), v \neq 0} R(v). \]
Let $\Omega_1 \subset \Omega_2$.

We will write $\{\lambda_k(\Omega_i)\}$, $i = 1, 2$ for the sequence of eigenvalues of $\Omega_i$, $i = 1, 2$. It follows from the variational characterization that for each $k \in \mathbb{N}$,

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This is because extension of a function by zero outside $\Omega_1$ gives an imbedding of $H^1_0(\Omega_1)$ into $H^1_0(\Omega_2)$. 
The first eigenfunction

An important property of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$) is that if $u$ is in that space, then so are $u^+$ and $u^-$. So we can use these as test functions in the weak formulation:

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega} u v \, dx.$$
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Setting $u = u_1$, $\lambda = \lambda_1$ and $v = u^{\pm}$, we easily deduce that

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Since $\lambda_1$ is the absolute minimum of $R(v)$ for $v \in H^1_0(\Omega)$, we deduce that $u_1^{\pm}$ are eigenfunctions corresponding to $\lambda_1$ as well. By the strong maximum principle for the Laplacian, it follows that $u_1^{\pm} \equiv 0$ or $u_1^{\pm} > 0$ in all of $\Omega$. Since $u_1 \neq 0$, both cannot be simultaneously zero, nor can both be simultaneously strictly positive over all of $\Omega$. Thus,

$$u_1 = u_1^+ \text{ or } u_1^- \text{ in } \Omega.$$
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Since $u_i$ are all orthogonal to $u_1$ (in $L^2(\Omega)$) for $i \geq 2$, it follows that they all change sign inside $\Omega$.

It also follows that $\lambda_1$ is a \textit{simple} eigenvalue.
If $u$ is an eigenfunction, then a nodal domain of $u$ is a subdomain of $\Omega$ where $u$ has a constant sign.
Nodal Domains

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**Theorem**

*(Courant):* Let $k \geq 2$. Then $u_k$ can have at most $k$ nodal domains.

**Corollary:** If $k = 2$, then $u_2$ has exactly two nodal domains.
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**Theorem**

(Pleijel): There exists a positive integer \( k_0 \) such that for all \( k \geq k_0 \), the number of nodal domains of an eigenfunction of \( \lambda_k \) is strictly less than \( k \).
When $\Omega \subset \mathbb{R}^2$ is a convex domain, then the curve

$$\{ x \in \overline{\Omega} : u(x) = 0 \},$$

called the nodal line, hits $\partial \Omega$ exactly at two points.
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**Conjecture**

(Payne) The same is true for any simply connected plane domain.

This is still open.
Asymptotic Behaviour

$\Omega \subset \mathbb{R}^N$. Let $|\Omega|$ denote the (N-dimensional) Lebesgue measure of $\Omega$. Weyl’s asymptotic Formula

$$\lambda_k(\Omega) \sim 4\pi^2 \left( \frac{k}{\omega_N |\Omega|} \right)^{\frac{2}{N}}$$

as $k \to \infty$, where $\omega_N$ is the volume of the unit ball in $\mathbb{R}^N$. 
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\[
\omega_N = \frac{\pi^\frac{N}{2}}{\Gamma(\frac{N}{2} + 1)}.
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\[ \omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2} + 1\right)}. \]

Pleijel (\(N = 2\)):

\[ \sum_{k=1}^{\infty} e^{-\lambda_k(\Omega)t} \sim \frac{A}{2\pi t} - \frac{L}{4} \frac{1}{\sqrt{2\pi t}} \]

as \(t \to 0\), where \(A\) is the area and \(L\) is the perimeter of \(\Omega \subset \mathbb{R}^2\).
Let $\Omega_i \subset \mathbb{R}^N$, $i = 1, 2$. We say that $\Omega_1$ and $\Omega_2$ are isospectral if

$$\lambda_k(\Omega_1) = \lambda_k(\Omega_2)$$

for all $k \geq 1$. 

**Question (Kac, 1966)**

If $\Omega_1$ and $\Omega_2$ are isospectral, then are they isometric as well? i.e. Can one be obtained from the other by a translation and rotation? (‘Can one hear the shape of a drum?’)

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Example of isospectral domains
Let $N = 2$ and let $\Omega_1$ be a disc. If $\Omega_i$, $i = 1, 2$ are isospectral, then they have the same area, $A$ and the same perimeter, $L$. But then, since $\Omega_1$ is a disc, we have $L^2 = 4\pi A$, which is now true for $\Omega_2$ as well and so, by the classical isoperimetric inequality, $\Omega_2$ has to be a disc of the same size as well.
Let $\Omega \subset \mathbb{R}^N$. let $\Omega^*$ be the ball with centre at the origin and such that $|\Omega^*| = |\Omega|$.

Let $u : \Omega \to \mathbb{R}$ be an integrable function.

$u^\# : [0, |\Omega|] \to \mathbb{R}$ is its one-dimensional decreasing rearrangement.
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If $\mu(t) = |u > t|$ is the distribution function of $u$, then, roughly, $u^\#$ is the inverse function. The Schwarz symmetrization of $u$ is $u^* : \Omega^* \rightarrow \mathbb{R}$ defined by

$$u^*(r) = u^\#(\omega_N r^N)$$

where $r^2 = \sum_{i=1}^{N} |x_i|^2$, $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$.

Thus, $u^*$ is a radial and radially decreasing function.
- $u$, $u^\#$, and $u^*$ are equimeasurable, i.e. they have the same distribution function.

- If $F: \mathbb{R} \to \mathbb{R}$ is a non-negative Borel function, then
  \[
  \int_{\Omega} F(u) \, dx = \int_{\Omega^*} F(u^*) \, dx.
  \]
  In particular, all $L^p$-norms of $u$ and $u^*$ are the same.

- **Hardy-Littlewood Inequality:**
  \[
  \int_{\Omega} uv \, dx \leq \int_{\Omega^*} u^* v^* \, dx.
  \]

- **Polya-Szegö Inequality:** If $u \in H^1_0(\Omega)$ and if $u \geq 0$ in $\Omega$, then $u^* \in H^1_0(\Omega^*)$ and
  \[
  \int_{\Omega} |\nabla u|^2 \, dx \geq \int_{\Omega^*} |\nabla u^*|^2 \, dx.
  \]
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- Polya-Szegö Inequality: if \( u \in H^1_0(\Omega) \) and if \( u \geq 0 \) in \( \Omega \), then \( u^* \in H^1_0(\Omega^*) \) and
  \[
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Rayleigh-Faber-Krahn
Of all domains of fixed measure, the ball has the least first eigenvalue.

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Consequence: In any dimension, given two isospectral domains, one of them being a ball, the other is also a ball.

Proof: Since they are isospectral, by Weyl’s formula, they have the same measure. Thus we can consider them as \( \Omega \) and \( \Omega^* \). Now, by the equality of \( \lambda_1 \), it follows that \( \Omega \) is also a ball.
Proof of the inequality:
Let \( u_1 \) be an eigenfunction corresponding to \( \lambda_1(\Omega) \). Then \( u_1 \in H^1_0(\Omega) \) and \( u_1 > 0 \) in \( \Omega \). So \( u_1^* \in H^1_0(\Omega^*) \) and

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Thus,

\[
\lambda_1(\Omega) = R_\Omega(u_1) \geq R_{\Omega^*}(u_1^*) \geq \lambda_1(\Omega^*).
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Consider the Neumann problem:

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Szegö - Weinberger:

\[\mu_2(\Omega) \leq \mu_2(\Omega^*).\]
Krahn-Szegö: Amongst all domains of fixed measure, $c$, the minimiser of $\lambda_2$ is the disjoint union of two balls, each of measure $c/2$. 
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**Theorem**

Let \( c > 0 \) and let \( k \) be a positive integer. There exists a convex domain \( \tilde{\Omega} \) such that \( |\tilde{\Omega}| = c \) and

\[
\lambda_k(\tilde{\Omega}) = \min \left\{ \lambda_k(\Omega) : \Omega \subset \mathbb{R}^N, \text{\( \Omega \) is convex, } |\Omega| = c \right\}.
\]
Other Dirichlet eigenvalues

Krahn-Szegö: Amongst all domains of fixed measure, $c$, the minimiser of $\lambda_2$ is the disjoint union of two balls, each of measure $c/2$. There is no minimizer for $\lambda_2$ amongst connected sets of fixed measure.

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**Open Problem**

Find the shape of the convex minimizer of $\lambda_2$?
Theorem

(Bucur-Henrot) There exists a minimiser for $\lambda_3$ amongst domains of fixed measure.
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Open Problem

Prove that the minimiser for $\lambda_3$ is a ball for dimensions $N = 2, 3$ and is the disjoint union of three identical balls for dimensions $N \geq 4$. 

Theorem (Bucur, 2012) For every $k \in \mathbb{N}$, there exists a minimiser for $\lambda_k$ amongst domains of equal measure.

Conjecture (Szegö): The minimiser of $\lambda_k$ is a ball or is a disjoint union of balls.

Answer: No. (Wolf-Keller) The 13th eigenvalue of a square is strictly less than that of any union of discs of equal area, in the plane. Numerical experiments show that for $k \geq 5$, the minimiser is not a ball, nor a disjoint union of balls, in the plane.

Open Problem

Amongst domains of fixed measure in $\mathbb{R}^N$, the $N$-ball minimises $\lambda_{N+1}$. 

S. Kesavan (IMSc)
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Amongst domains of fixed measure in $\mathbb{R}^N$, the $N$-ball minimises $\lambda_{N+1}$. 
Payne-Polya-Weinberger Conjecture (1950s): $N = 2$

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\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(\Omega^*)}{\lambda_1(\Omega^*)}.
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Clever use of Schwarz symmetrization and properties of Bessel functions.
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Other conjectures, still open, due to Payne-Polya:

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Other Problems

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$$\frac{\lambda_{m+1}(\Omega)}{\lambda_m(\Omega)} \leq \frac{\lambda_2(\Omega^*)}{\lambda_1(\Omega^*)}.$$
Vibration of a clamped plate

\[ \Delta^2 u = \Lambda u \text{ in } \Omega, \]
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The buckling of a clamped plate

\[ \Delta^2 u = -\sigma \Delta u \text{ in } \Omega, \]
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The existence of a sequence of eigenvalues and a complete orthonormal sequence of eigenfunctions follows as before. However,

- \( \Lambda_1 \) and \( \sigma_1 \) are not necessarily simple eigenvalues (but true for a ball).
- The first eigenfunction in either case is not necessarily of constant sign in \( \Omega \) (but true for a ball).
Conjecture (Rayleigh, 1894):

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Rayleigh’s conjecture proved for \( N = 2 \) by Nadirashvili (1992) and for \( N = 2, 3 \) by Ashbaugh and Benguria. Case of general \( N \) is open. Polya-Szegö conjecture still open in all dimensions. Both are easy to prove if we know that the first eigenfunction does not change sign, but this is unfortunately not true!
We can show that

\[ \Lambda_1(\Omega) \geq c \Lambda_1(\Omega^*) \]

and

\[ \sigma_1(\Omega) \geq d \sigma_1(\Omega^*) \]

with \( c = d = 1/2 \).
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It has been shown that these inequalities hold in \( \mathbb{R}^N \) with \( c = c_N \) and \( d = d_N \) where \( 0 < c_N, d_N < 1 \) and \( c_N, d_N \) are computable constants which tend to unity as \( N \to \infty \).
Let $1 < p < \infty$. Consider the nonlinear eigenvalue problem:

$$-\text{div}(|\nabla u|^{p-2} \nabla u) = \lambda |u|^{p-2} u \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial \Omega.$$
The $p$-Laplacian

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Weak form: Find $\lambda \in \mathbb{R}$ and $u \in W^{1,p}_0(\Omega)$, $u \not\equiv 0$, such that, for every $v \in C^\infty_c(\Omega)$,

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Weak form: Find $\lambda \in \mathbb{R}$ and $u \in W^{1,p}_0(\Omega)$, $u \neq 0$, such that, for every $v \in C_c^\infty(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2} \nabla u . \nabla v \, dx = \int_\Omega |u|^{p-2} uv \, dx,$$

where $W^{1,p}_0(\Omega)$ is the Sobolev space which is the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,p} = \left( \int_\Omega (|\nabla u|^p + |u|^p) \, dx \right)^{\frac{1}{p}}.$$
As in the case $p = 2$, by Poincaré’s inequality, the norm

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is an equivalent norm. This is a nonlinear problem and so we do not have an eigenspace attached to an eigenvalue. The eigenvalues are critical values of the Rayleigh quotient

$$ R_p(u) = \frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\Omega} |u|^p \, dx}. $$

The minimum of $R_p(u)$ is attained and so it is called the principal eigenvalue and the minimiser is an eigenfunction. It can be shown that all eigenfunctions associated to the principal eigenvalue are scalar multiples of each other and so we say that this eigenvalue, called $\lambda_1$, is 'simple'. Further, the eigenfunctions do not change sign in $\Omega$. $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$ and the principal eigenfunction in a ball is radial.
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$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$$

and the principal eigenfunction in a ball is radial.
Using critical point theory (Lusternik-Schnirelman) applied to the Rayleigh quotient, we can show the existence of an increasing sequence of positive eigenvalues, which tends to infinity.
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Open Problem

Are these the only eigenvalues?

It can be shown that there are no other eigenvalues between $\lambda_1$ and $\lambda_2$. 
Thank You!