On the spectrum of the Laplacian

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and $\partial\Omega$ denotes the boundary of Ω.
$N = 1$: vibrating string which is fixed at both ends.
$N = 2$: vibrating membrane (drum) fixed along the boundary.

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We get that the only non-trivial solutions are

$$\lambda_n = n^2\pi^2, \ u_n \ = \ C \sin n\pi x, \ n \in \mathbb{N}$$

where C is any real constant. If we fix $C = \sqrt{2}$, we get

$$\int_0^1 u_n^2(x) \, dx \ = \ 1, \text{ for all } n \in \mathbb{N}.$$

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- Notice that

$$\int_0^1 u_n(x)u_m(x)\ dx \ = \ \delta_{nm}$$

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- $u_1 > 0$ in $(0,1)$.
- u_n has exactly $n-1$ zeros in $(0,1)$.

Sobolev Spaces

Let $\Omega \subset \mathbb{R}^N$ be a reasonably smooth domain. Then $H^1(\Omega)$ is the completion of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,\Omega} \; = \; \left[\int_\Omega \left(|\nabla u(x)|^2 + |u(x)|^2\right) \; dx\right]^{\frac{1}{2}}$$

where $x = (x_1, x_2, \cdots, x_N)$ and $dx = dx_1 dx_2 \cdots dx_N$.

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Poincaré's Inequality states that the semi-norm

$$|u|_{1,\Omega} \;=\; \left[\int_\Omega |\nabla u(x)|^2 \; dx\right]^{\frac{1}{2}}$$

is also a norm for $H_0^1(\Omega)$, which is equivalent to the norm $\|u\|_{1,\Omega}$.

Equivalently, $H^1(\Omega)$ can be thought of as the space of $L^2(\Omega)$ 'functions' whose distributional derivatives of the first order are also in $L^2(\Omega)$ and $H_0^1(\Omega)$ is the closed subspace of 'functions' in $H^1(\Omega)$ which 'vanish' on the boundary $\partial\Omega$.

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Its weak form is to find $v \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla v.\nabla w \; dx \; = \; \int_\Omega fw$$

for all $w \in H_0^1(\Omega)$. This problem has a unique solution and we define $G : L^2(\Omega) \to H_0^1(\Omega)$ by $G(f) = v$. This is a continous operator and if we compose it with the inclusion

$$H_0^1(\Omega) \; \subset \; L^2(\Omega)$$

which is compact (Rellich's theorem) we get that G is a compact operator of $L^2(\Omega)$ into itself and it is easy to see that it is self-adjoint as well.

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$$u = G(\lambda u)$$

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Let us write

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty$$

with the λ_n being repeated as many times as the dimension of the corresponding eigenspace.

Example

$\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$. Then, it is easy to see that $\lambda_{nm} = \pi^2(n^2 + m^2)$ is an eigenvalue with corresponding eigenfunction

$$u_{nm} \ = \ 2 \sin n\pi x \sin m\pi y.$$

That these are the only ones needs proof and follows from the fact that $\{u_{nm}\}$ is a complete orthonormal basis for $L^2(\Omega)$. Thus, $\lambda_1 = 2\pi^2$ while $\lambda_2 = \lambda_3 = 5\pi^2$ corresponding to $n = 1, m = 2$ and $n = 2, m = 1$ and the space of eigenfunctions is two dimensional spanned by $2 \sin \pi x \sin 2\pi y$ and $2 \sin 2\pi x \sin \pi y$.

Example

Ω is the unit disc in $\mathbb{R}^2$. In polar coordinates, we have

$$-\left[u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right] = \lambda u.$$

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We look for solutions of the form $u(r,\theta) = v(r)w(\theta)$ and this leads us to look at

$$w'' + kw \;=\; 0,\; w \text{ is } 2\pi - \mathrm{periodic}$$

and

$$v'' + \frac{1}{r}v' + \left(\lambda - \frac{k}{r^2}\right) \;=\; 0$$

with $v'(0) = v(1) = 0$. The first equation implies that $k = n^2, n \in \{0\} \cup \mathbb{N}$ and substituting it in the second leads us to the Bessel's equation. In particular, u_1 corresponds to $k = 0$ and is a radial function and λ_1 comes from the first zero of the Bessel function J_0:

$$\lambda_1 \;=\; j_{0,1}^2,\; u_1 \;=\; CJ_0(j_{0,1}r).$$

If $j_{0,l}$ is the l-th zero of J_0, then $j_{0,l}^2$ is a simple eigenvalue with eigenfunction $CJ_0(j_{0,l}r)$ which is also radial. While u_1 is positive in Ω, the others change sign.

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If we take $k = n^2, n \in \mathbb{N}$, then for $n, l \geq 1$ we have the double eigenvalue $j_{n,l}^2$ with eigenspace spanned by

$$CJ_n(j_{n,l}r)\cos n\theta, \text{ and } CJ_n(j_{n,l}r)\sin n\theta.$$

Variational Characterization

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Given an orthonormal basis of eigenfunctions $\{u_n\}$ corresponding to the eigenvalues $\{\lambda_n\}$ listed in increasing order, taking into account the multiplicity, set, for $k \in \mathbb{N}$,

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Then, for $k \in \mathbb{N}$,

$$\begin{aligned} \lambda_k &= R(u_k) \\ &= \max_{v \in V_k, v \neq 0} R(v) \\ &= \min_{v \perp V_{k-1}, v \neq 0} R(v) \\ &= \min_{V \subset H_0^1(\Omega), \dim V = k} \max_{v \in V, v \neq 0} R(v) \end{aligned}$$

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In particular,

$$\lambda_1 \ = \ \min_{v \in H_0^1(\Omega), v \neq 0} R(v).$$

Monotonicity with respect to domain inclusion

Let $\Omega_1 \subset \Omega_2$.
We will write $\{\lambda_k(\Omega_i)\}, i = 1, 2$ for the sequence of eigenvalues of $\Omega_i, i = 1, 2$. It follows from the variational characterization that for each $k \in \mathbb{N}$,

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This is because extension of a function by zero outside Ω_1 gives an imbedding of $H_0^1(\Omega_1)$ into $H_0^1(\Omega_2)$.

The first eigenfunction

An important property of $H^1(\Omega)$ (resp. $H_0^1(\Omega)$) is that if u is in that space, then so are u^+ and u^-. So we can use these as test functions in the weak formulation:

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Since λ_1 is the absolute minimum of $R(v)$ for $v \in H_0^1(\Omega)$, we deduce that $u_1^\pm$ are eigenfunctions corresponding to λ_1 as well. By the strong maximum principle for the Laplacian, it follows that $u_1^\pm \equiv 0$ or $u_1^\pm > 0$ in all of Ω. Since $u_1 \not\equiv 0$, both cannot be simultaneously zero, nor can both be simultaneously strictly positive over all of Ω. Thus,

$$u_1 \; = \; u_1^+ \text{ or } u_1^- \text{ in } \Omega.$$

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It also follows that λ_1 is a **simple** eigenvalue.

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Theorem

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Corollary: If $k = 2$, then u_2 has exactly two nodal domains.

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Theorem

(Pleijel): There exists a positive integer k_0 such that for all $k \geq k_0$, the number of nodal domains of an eigenfunction of λ_k is strictly less than k.

When $\Omega \subset \mathbb{R}^2$ is a convex domain, then the curve

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called the nodal line, hits $\partial\Omega$ exactly at two points.

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Conjecture

(Payne) The same is true for any simply connected plane domain.

This is still open.

Asymptotic Behaviour

$\Omega \subset \mathbb{R}^N$. Let $|\Omega|$ denote the (N-dimensional) Lebesgue measure of Ω. Weyl's asymptotic Formula

$$\lambda_k(\Omega) \sim 4\pi^2 \left(\frac{k}{\omega_N |\Omega|}\right)^{\frac{2}{N}}$$

as $k \to \infty$, where ω_N is the volume of the unit ball in $\mathbb{R}^N$.

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$$\omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}.$$

Pleijel ($N = 2$):

$$\sum_{k=1}^{\infty} e^{-\lambda_k(\Omega)t} \sim \frac{A}{2\pi t} - \frac{L}{4}\frac{1}{\sqrt{2\pi t}}$$

as $t \to 0$, where A is the area and L is the perimeter of $\Omega \subset \mathbb{R}^2$.

Isospectral Domains

Let $\Omega_i \subset \mathbb{R}^N, \ i = 1, 2.$
We say that Ω_1 and Ω_2 are isospectral if

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Question (Kac, 1966)

If Ω_1 and Ω_2 are isospectral, then are they isometric as well? *i.e.* Can one be obtained from the other by a translation and rotation?('Can one hear the shape of a drum?')

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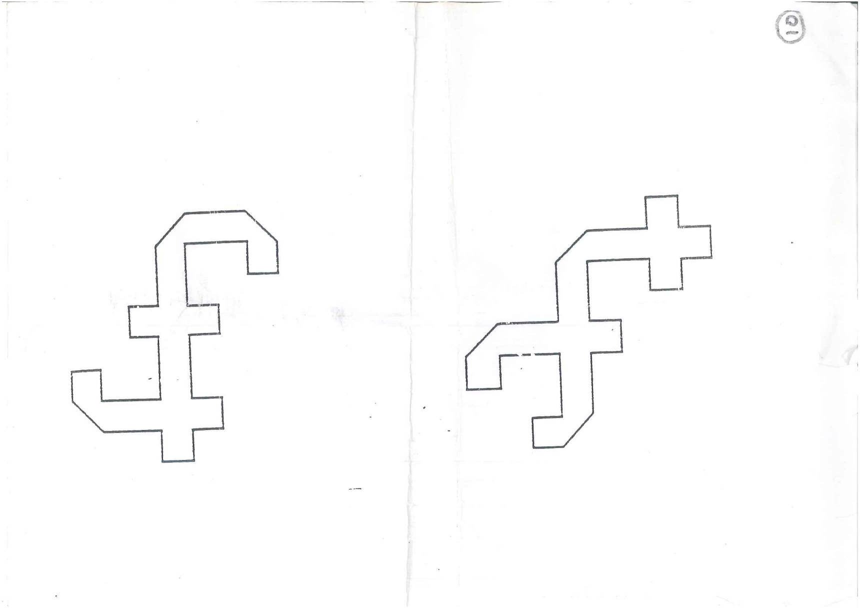
Answer: 'No'. The case $N = 2$ resisted resolution till the early 90's.
(Gordon, C., Webb, D. L. and Wolpert, S., BAMS (New Series),Vol. 27, No. 1, pp.134-138, 1992).
Can produce examples by paper folding: See, Chapman, S. J.: Drums that sound the same, AMM,102, Feb. 1995,pp.124-138.

Example of isospectral domains



Let $N = 2$ and let Ω_1 be a disc. If $\Omega_i,\ i = 1, 2$ are isopsectral, then they have the same area, A and the same perimeter, L. But then, since Ω_1 is a disc, we have $L^2 = 4\pi A$, which is now true for Ω_2 as well and so, by the classical isoperimetric inequality, Ω_2 has to be a disc of the same size as well.

Schwarz Symmetrization

Let $\Omega \subset \mathbb{R}^N$. let Ω^* be the ball with centre at the origin and such that $|\Omega^*| = |\Omega|$.
Let $u : \Omega \to \mathbb{R}$ be an integrable function.
$u^\# : [0, |\Omega|] \to \mathbb{R}$ is its **one-dimensional decreasing rearrangement**.

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If $\mu(t) = |u > t|$ is the distribution function of u, then, roughly, $u^\#$ is the inverse function. The Schwarz symmetrization of u is $u^* : \Omega^* \to \mathbb{R}$ defined by

$$u^*(r) \; = \; u^\#(\omega_N r^N)$$

where $r^2 = \sum_{i=1}^N |x_i|^2, \; x = (x_1, \cdots, x_N) \in \mathbb{R}^N$.
Thus, u^* is a radial and radially decreasing function.

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- Polya-Szegö Inequality: if $u \in H_0^1(\Omega)$ and if $u \geq 0$ in Ω, then $u^* \in H_0^1(\Omega^*)$ and

$$\int_{\Omega} |\nabla u|^2\ dx \ \geq \ \int_{\Omega^*} |\nabla u^*|^2\ dx.$$

Extremum Problems for the eigenvalues

Rayleigh-Faber-Krahn

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Consequence: In any dimension, given two isospectral domains, one of them being a ball, the other is also a ball.
Proof: Since they are isospectral, by Weyl's formula, they have the same measure. Thus we can consider them as Ω and Ω^*. Now, by the equality of λ_1, it follows that Ω is also a ball.

Proof of the inequality:
Let u_1 be an eigenfunction corresponding to $\lambda_1(\Omega)$. Then $u_1 \in H_0^1(\Omega)$ and $u_1 > 0$ in Ω. So $u_1^* \in H_0^1(\Omega^*)$ and

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Thus,

$$\lambda_1(\Omega) \; = \; R_\Omega(u_1) \; \geq \; R_{\Omega^*}(u_1^*) \; \geq \; \lambda_1(\Omega^*).$$

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Szegö - Weinberger:

$$\mu_2(\Omega) \leq \mu_2(\Omega^*).$$

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Theorem

Let $c > 0$ and let k be a positive integer. There exists a convex domain $\widetilde{\Omega}$ such that $|\widetilde{\Omega}| = c$ and

$$\lambda_k(\widetilde{\Omega}) \; = \; \min \left\{ \lambda_k(\Omega) \; : \; \begin{array}{l} \Omega \subset \mathbb{R}^N, \\ \Omega \text{ is convex}, |\Omega| = c \end{array} \right\}.$$

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Open Problem

Find the shape of the convex minimizer of λ_2?

The third eigenvalue

Theorem

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(Wolf-Keller) In the plane, the ball is a local minimiser for λ_3.

Open Problem

Prove that the minimiser for λ_3 is a ball for dimensions $N = 2, 3$ and is the disjoint union of three identical balls for dimensions $N \geq 4$.

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Answer: No.
(Wolf-Keller) The 13th eigenvalue of a square is strictly less than that of any union of discs of equal area, in the plane.
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Open Problem

Amongst domains of fixed measure in $\mathbb{R}^N$, the N-ball minimises λ_{N+1}.

Other Problems

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$$\frac{\lambda_{m+1}(\Omega)}{\lambda_m(\Omega)} \leq \frac{\lambda_2(\Omega^*)}{\lambda_1(\Omega^*)}.$$

The Biharmonic Operator

Vibration of a clamped plate

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- Λ_1 and σ_1 are not necessarily simple eigenvalues (but true for a ball).
- The first eigenfunction in either case is not necessarily of constant sign in Ω (but true for a ball).

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Rayleigh's conjecture proved for $N = 2$ by Nadirashvili (1992) and for $N = 2, 3$ by Ashbaugh and Benguria. Case of general N is open.
Polya-Szegö conjecture still open in all dimensions.
Both are easy to prove if we know that the first eigenfunction does not change sign, but this is unfortunately not true!

We can show that

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It has been shown that these inequalities hold in $\mathbb{R}^N$ with $c = c_N$ and $d = d_N$ where $0 < c_N, d_N < 1$ and c_N, d_N are computable constants which tend to unity as $N \to \infty$.

The p-Laplacian

Let $1 < p < \infty$. Consider the nonlinear eigenvalue problem:

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Weak form: Find $\lambda \in \mathbb{R}$ and $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$, such that, for every $v \in C_c^\infty(\Omega)$,

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where $W_0^{1,p}(\Omega)$ is the Sobolev space which is the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,p} \; = \; \left(\int_\Omega (|\nabla u|^p + |u|^p) \; dx \right)^{\frac{1}{p}}.$$

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This is a nonlinear problem and so we do not have an eigenspace attached to an eigenvalue. The eigenvalues are critical values of the Rayleigh quotient

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The minimum of R_p is attained and so it is called the principal eigenvalue and the minimiser is an eigenfunction. It can be shown that all eigenfunctions associated to the principal eigenvalue are scalar multiples of each other and so we say that this eigenvalue, called λ_1, is 'simple'.

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$$\lambda_1(\Omega) \ \geq \ \lambda_1(\Omega^*)$$

and the principal eigenfunction in a ball is radial.

Using critical point theory (Lusternik-Schnirelman) applied to the Rayleigh quotient, we can show the existence of an increasing sequence of positive eigenvalues, which tends to infinity.

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Open Problem

Are these the only eigenvalues?

It can be shown that there are no other eigenvalues between λ_1 and λ_2.

Thank You!