TWO WORKED OUT EXAMPLES OF ROTATIONS USING QUATERNIONS

This note is an attachment to the article “Rotations and Quaternions” which in turn is a companion to the video of the talk by the same title.

Example 1. Determine the image of the point (1, -1, 2) under the rotation by an angle of 60° about an axis in the $yz$-plane that is inclined at an angle of 60° to the positive $y$-axis.

**Solution:** The unit vector $u$ in the direction of the axis of rotation is $\cos 60° j + \sin 60° k = \frac{1}{2} j + \frac{\sqrt{3}}{2} k$. The quaternion (or vector) corresponding to the point $p = (1, -1, 2)$ is of course $p = i - j + 2k$. To find the image of $p$ under the rotation, we calculate $qpq^{-1}$ where $q$ is the quaternion $\cos \frac{\theta}{2} + \sin \frac{\theta}{2} u$ and $\theta$ the angle of rotation (60° in this case). The resulting quaternion—if we did the calculation right—would have no constant term and therefore we can interpret it as a vector. That vector gives us the answer.

We have $q = \frac{\sqrt{3}}{2} + \frac{1}{2} i = \frac{\sqrt{3}}{2} i + \frac{\sqrt{3}}{2} k = \frac{1}{2} (2\sqrt{3} j + \sqrt{3} k)$. Since $q$ is by construction a unit quaternion, its inverse is its conjugate: $q^{-1} = \frac{1}{2} (2\sqrt{3} j - \sqrt{3} k)$. Now, computing $qp$ in the routine way, we get

$$qp = \frac{1}{4} ((1 - 2\sqrt{3}) + (2 + 3\sqrt{3}) i - \sqrt{3} j + (4\sqrt{3} - 1) k)$$

and then another long but routine computation gives

$$qpq^{-1} = \frac{1}{8} ((10 + 4\sqrt{3}) i + (1 + 2\sqrt{3}) j + (14 - 3\sqrt{3}) k)$$

The point corresponding to the vector on the right hand side in the above equation is the image of $(1, -1, 2)$ under the given rotation. Explicitly, that point is

$$\left( \frac{10 + 4\sqrt{3}}{8}, \frac{1 + 2\sqrt{3}}{8}, \frac{14 - 3\sqrt{3}}{8} \right)$$

Example 2. Consider the rotation by 60° about an axis in the $xz$-plane inclined at an angle of 60° to the positive $x$-axis. Determine the composition of this rotation with that in the earlier example: the one in the earlier example acts first.

**Solution:** We need only compute $q'q$ where $q$ is as in the previous example and $q' = \cos 30° + \sin 30° v$ where $v$ is the unit vector in the direction of the axis of the rotation specified in this example. We have $v = \cos 60° i + \sin 60° k = \frac{1}{2} (i + \sqrt{3} k)$. So $q' = \frac{\sqrt{3}}{2} + \frac{1}{2} (i + \sqrt{3} k) = \frac{1}{2} (2\sqrt{3} i + j + \sqrt{3} k)$. Now a long but routine calculation gives:

$$q'q = \frac{1}{4} (2\sqrt{3} + i + \sqrt{3} k) \cdot \frac{1}{4} (2\sqrt{3} + j + \sqrt{3} k) = \frac{1}{16} (9 + \sqrt{3} i + \sqrt{3} j + 13 k)$$

We now need to determine an angle $\theta$ and a unit vector $v$ so that

$$\cos \frac{\theta}{2} + \sin \frac{\theta}{2} v = \frac{1}{16} (9 + \sqrt{3} i + \sqrt{3} j + 13 k)$$

Let us assume $0 \leq \frac{\theta}{2} \leq 180°$, so that the value of $\cos \frac{\theta}{2}$ (which is $\frac{9}{16}$ in our case) determines $\frac{\theta}{2}$. Then $\sin \frac{\theta}{2}$ is non-negative and in the present case we get $\sin \frac{\theta}{2} = \frac{\sqrt{7}}{16}$. Now we determine $v$:

$$v = \frac{1}{\sin \frac{\theta}{2}} \cdot \frac{1}{16} (\sqrt{3} i + \sqrt{3} j + 13 k) = \frac{1}{5\sqrt{7}} (\sqrt{3} i + \sqrt{3} j + 13 k) = \frac{1}{35} (\sqrt{21} i + \sqrt{21} j + 13 \sqrt{7} k)$$

The composite rotation is by the angle $\theta$ around the axis determined by the unit vector $v$. \[\square\]
The association $p \mapsto qpq^{-1}$, where $p$ is a point in 3-space interpreted as a vector and $q$ a unit quaternion, is what enables us to attach rotations to unit quaternions. Since the association does not change if we replace $q$ by $-q$, it follows that both $q$ and $-q$ are attached to the same rotation. Moreover, $q$ and $-q$ are the only ones that have the same association. Indeed, if $qpq^{-1} = q'pq'^{-1}$ for all $p$, then $qq'^{-1}$ commutes with all quaternions, and so has no imaginary component; being a unit quaternion as well, it can only be $\pm 1$; thus $q' = \pm q$. 