Snapshots from Transformation Geometry

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What is geometry?

There are many different ways of defining ‘geometry’ but one of them is:

Geometry is the study of shapes, and how their properties are affected by given groups of transformations: which properties are left unaltered, and which ones undergo a change.

This view of geometry is due to the mathematician Felix Klein (1849–1925).

What is a 'Geometric Transformation'?

A transformation of the plane is a function defined on the plane, moving points around according to a definite law.

Matters of interest: Is the function 'well behaved'? Is it smooth? Does it preserve length? Angles? Orientation? Area?

In today's talk we shall see how the use of transformations can give rise to elegant proofs of some geometrical propositions.

Affine maps

Let $\boldsymbol{f}$ be a bijection of the plane. We say that $\boldsymbol{f}$ is affine if it preserves the property of collinearity. Let the images of points $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ under $\boldsymbol{f}$ be $\boldsymbol{A}', \boldsymbol{B}', \boldsymbol{C}', \ldots$. Let the images of lines $\boldsymbol{l}, \boldsymbol{m}$ under $\boldsymbol{f}$ be $\boldsymbol{l}', \boldsymbol{m}'$. Then:

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- $\boldsymbol{l} \parallel \boldsymbol{m} \iff \boldsymbol{l}' \parallel \boldsymbol{m}'$

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Let f be a bijection of the plane. We say that f is affine if it preserves the property of collinearity. Let the images of points $A, B, C, \ldots$ under f be $A', B', C', \ldots$. Let the images of lines l, m under f be l', m'. Then:

- $l \parallel m \iff l' \parallel m'$
- B is the midpoint of $AC \iff B'$ is the midpoint of $A'C'$

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- $\boldsymbol{l \parallel m \iff l' \parallel m'}$
- $\boldsymbol{B}$ is the midpoint of $\boldsymbol{AC} \iff \boldsymbol{B'}$ is the midpoint of $\boldsymbol{A'C'}$
- $\boldsymbol{A, B, C}$ collinear $\implies \boldsymbol{AB : BC = A'B' : B'C'}$

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- $l \parallel m \iff l' \parallel m'$
- B is the midpoint of $AC \iff B'$ is the midpoint of $A'C'$
- A, B, C collinear $\implies AB : BC = A'B' : B'C'$
- Interior of $\triangle ABC$ is mapped to interior of $\triangle A'B'C'$

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Note the progression: congruence geometry, similarity geometry, affine geometry. This is in keeping with Klein's vision.

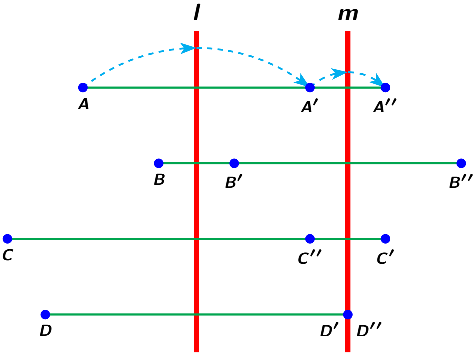
Notation

Symbol	Meaning
$\mathbf{T}_{PQ}$	Translation ('displacement') through vector $\boldsymbol{PQ}$
$\mathbf{H}_{P}$	Half-turn centred at point $\boldsymbol{P}$
$\mathbf{M}_{\ell}$	Mirror reflection in line $\boldsymbol{\ell}$
$\mathbf{R}_{P,\theta}$	Rotation centred at $\boldsymbol{P}$, through angle $\boldsymbol{\theta}$
$\mathbf{E}_{P,k}$	Enlargement centred at $\boldsymbol{P}$, with scale factor $\boldsymbol{k}$

Note: (i) $(\mathbf{T}_{PQ})^{-1} = \mathbf{T}_{QP}$ (ii) $\mathbf{H}_P$ and $\mathbf{M}_\ell$ are self-inverse (iii) inverse of $\mathbf{R}_{P,\theta}$ is $\mathbf{R}_{P,-\theta}$ (iv) inverse of $\mathbf{E}_{P,k}$ is $\mathbf{E}_{P,1/k}$ (v) $\mathbf{E}_{P,-1}$ is the same as $\mathbf{H}_P$

Composition of two reflections: parallel mirrors

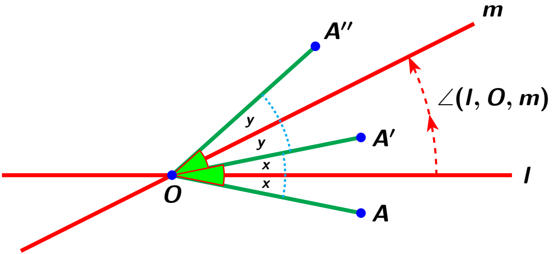
If $l \parallel m$, then $\mathbf{M}_l$ followed by $\mathbf{M}_m$ is equivalent to a displacement.



Segments AA'', BB'', CC'', DD'' have equal length: each is twice the distance between l & m.

Composition of two reflections: non-parallel mirrors

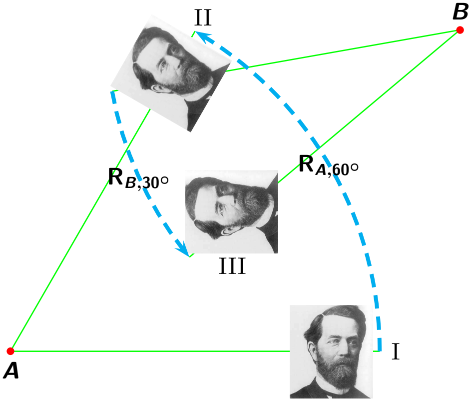
If $\neg(\boldsymbol{l} \parallel \boldsymbol{m})$, then $\mathbf{M}_{\boldsymbol{l}}$ followed by $\mathbf{M}_{\boldsymbol{m}}$ is equivalent to a rotation.



$\angle AOA'' = 2 \times \angle(l, O, m) =$ twice the directed angle from $\boldsymbol{l}$ to $\boldsymbol{m}$

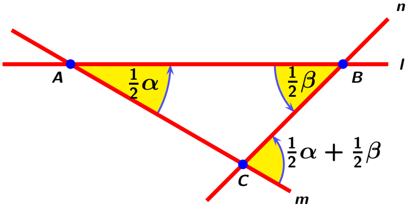
Composition of two rotations

(With due apologies to Herr Klein)



Here we see a motif rotated first about A by $60°$, then about B by $30°$. From the positions, it appears as though a single rotation could have taken the motif from I to III.

Locating the centre of $R_{B,\beta} \circ R_{A,\alpha}$



Draw line AB; draw lines m, n through A, B such that $\angle(m, l) = \frac{1}{2}\alpha$, $\angle(l, n) = \frac{1}{2}\beta$.

Keep directions in mind!

Let m, n meet at C. Then $\angle(m, n) = \frac{1}{2}\alpha + \frac{1}{2}\beta$. So:

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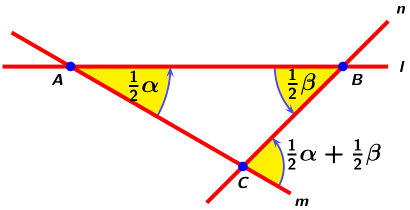

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$$\mathrm{R}_{A,\alpha} = \mathrm{M}_l \circ \mathrm{M}_m, \quad \mathrm{R}_{B,\beta} = \mathrm{M}_n \circ \mathrm{M}_l,$$

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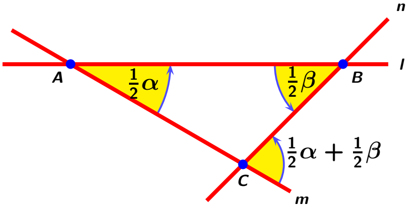

Draw line $\boldsymbol{AB}$; draw lines $\boldsymbol{m}, \boldsymbol{n}$ through $\boldsymbol{A}$, $\boldsymbol{B}$ such that $\angle(\boldsymbol{m}, \boldsymbol{l}) = \frac{1}{2}\alpha$, $\angle(\boldsymbol{l}, \boldsymbol{n}) = \frac{1}{2}\beta$.

Keep directions in mind!

Let $\boldsymbol{m}, \boldsymbol{n}$ meet at $\boldsymbol{C}$. Then $\angle(\boldsymbol{m}, \boldsymbol{n}) = \frac{1}{2}\alpha + \frac{1}{2}\beta$. So:

$$\mathsf{R}_{A,\alpha} = \mathsf{M}_l \circ \mathsf{M}_m, \quad \mathsf{R}_{B,\beta} = \mathsf{M}_n \circ \mathsf{M}_l,$$

$$\therefore \ \mathsf{R}_{B,\beta} \circ \mathsf{R}_{A,\alpha} = (\mathsf{M}_n \circ \mathsf{M}_l) \circ (\mathsf{M}_l \circ \mathsf{M}_m) \,.$$

So $\mathbf{R}_{B,\beta} \circ \mathbf{R}_{A,\alpha} = \mathbf{M}_n \circ (\mathbf{M}_l \circ \mathbf{M}_l) \circ \mathbf{M}_m = \mathbf{M}_n \circ \mathbf{M}_m$ and is therefore equivalent to the composite map $\mathbf{M}_n \circ \mathbf{M}_m$.

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But $\mathbf{M}_n \circ \mathbf{M}_m$ is equivalent to a rotation about the point where $\boldsymbol{m}$ and $\boldsymbol{n}$ meet, through twice $\angle(\boldsymbol{m}, \boldsymbol{n})$.

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Therefore, $\mathbf{R}_{B,\beta} \circ \mathbf{R}_{A,\alpha}$ is equivalent to the rotation $\mathbf{R}_{C,\alpha+\beta}$.

So $\mathbf{R}_{\boldsymbol{B},\beta} \circ \mathbf{R}_{\boldsymbol{A},\alpha} = \mathbf{M}_{\boldsymbol{n}} \circ (\mathbf{M}_{\boldsymbol{l}} \circ \mathbf{M}_{\boldsymbol{l}}) \circ \mathbf{M}_{\boldsymbol{m}} = \mathbf{M}_{\boldsymbol{n}} \circ \mathbf{M}_{\boldsymbol{m}}$ and is therefore equivalent to the composite map $\mathbf{M}_{\boldsymbol{n}} \circ \mathbf{M}_{\boldsymbol{m}}$.

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Therefore, $\mathbf{R}_{\boldsymbol{B},\beta} \circ \mathbf{R}_{\boldsymbol{A},\alpha}$ is equivalent to the rotation $\mathbf{R}_{\boldsymbol{C},\alpha+\beta}$.

Could anything go wrong with this analysis? Yes: it could happen that $\boldsymbol{m} \parallel \boldsymbol{n}$, in which case the lines $\boldsymbol{m}, \boldsymbol{n}$ do not meet at all!

This will happen if $\alpha + \beta$ is a multiple of $\mathbf{360}^\circ$.

However, the conclusion that $\mathbf{R}_{B,\beta} \circ \mathbf{R}_{A,\alpha} = \mathbf{M}_n \circ \mathbf{M}_m$ stays.

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But since $\boldsymbol{m} \parallel \boldsymbol{n}$, the map $\mathbf{M}_{\boldsymbol{n}} \circ \mathbf{M}_{\boldsymbol{m}}$ is a displacement.

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But since $\boldsymbol{m} \parallel \boldsymbol{n}$, the map $\mathbf{M}_{\boldsymbol{n}} \circ \mathbf{M}_{\boldsymbol{m}}$ is a displacement.

So if $\alpha + \beta$ is a multiple of $\mathbf{360}^\circ$, then $\mathbf{R}_{\boldsymbol{B},\beta} \circ \mathbf{R}_{\boldsymbol{A},\alpha}$ is a displacement.

(Counterintuitive? Or daily life wisdom?)

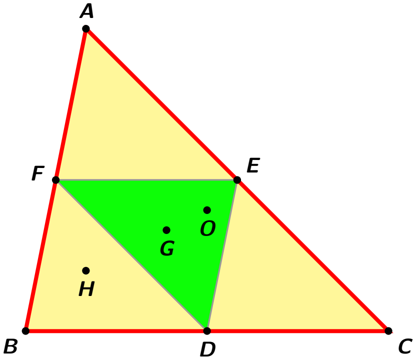
Part I

Problems and Theorems

We showcase some applications of the method of transformations.

One of Euler's (many) theorems

O: circumcentre, G: centroid, H: orthocentre; $\overrightarrow{OH} = 3\overrightarrow{OG}$



- $\longrightarrow$ $\triangle \boldsymbol{ABC}$, with circumcentre $\boldsymbol{O}$, centroid $\boldsymbol{G}$, orthocentre $\boldsymbol{H}$
- $\longrightarrow$ $\boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F}$: midpoints of sides
- $\longrightarrow$ Consider $\mathbf{E}_{G,-1/2}$:

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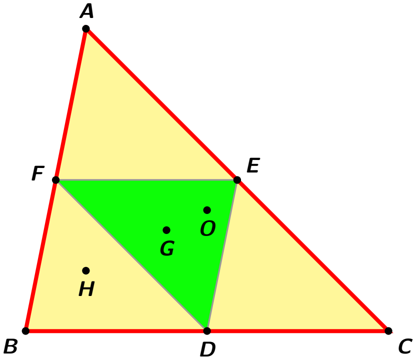

- $\rightarrow$ $\triangle ABC$, with circumcentre O, centroid G, orthocentre H
- $\rightarrow$ D, E, F: midpoints of sides
- $\rightarrow$ Consider $\mathbf{E}_{G,-1/2}$: it maps A, B, C to D, E, F.

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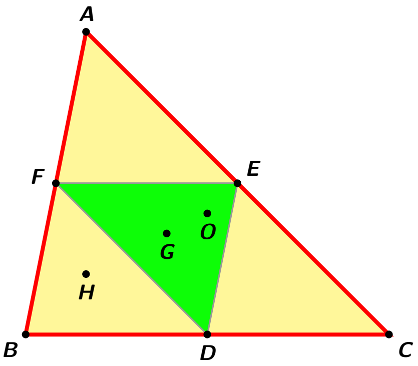

- $\triangle ABC$, with circumcentre O, centroid G, orthocentre H
- D, E, F: midpoints of sides
- Consider $\mathbf{E}_{G,-1/2}$: it maps A, B, C to D, E, F. It maps the perp^r to BC through A to the perp^r to EF through D.

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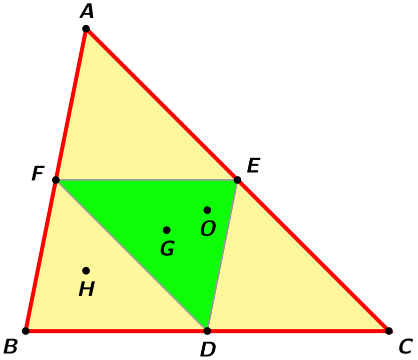

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One of Euler's (many) theorems

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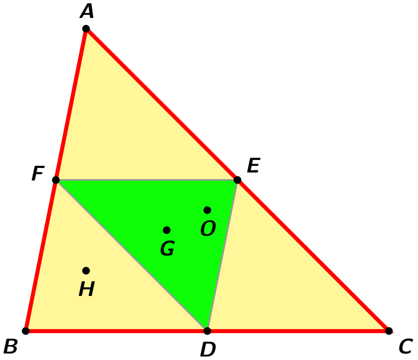

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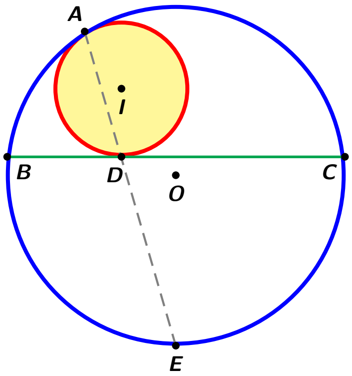
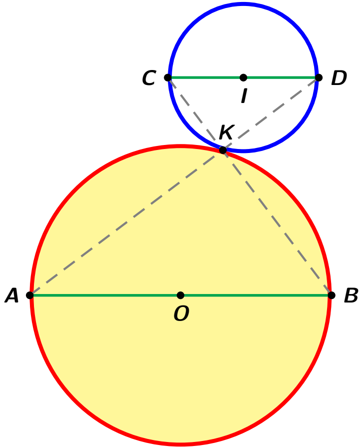
- Consider $\mathbf{E}_{G,-1/2}$: it maps A, B, C to D, E, F. It maps the perp^r to BC through A to the perp^r to EF through D. So it maps H to O. It follows that $\overrightarrow{OH} = 3\overrightarrow{OG}$.

Two tangent circles



- → Circles (I, A) and (O, A) touch internally at A.
- → Chord BC of (O, A) is tangent to (I, A) at D.
- → Point E lies on (O, A) such that $OE \perp BC$.
- → Points A, D, E lie in a straight line.

Two more tangent circles



→ Circles (I, K) and (O, K) touch each other at K.

→ AB and CD are a pair of parallel diameters of the two circles (labeled suitably)

→ Points B, K, C lie in a straight line, as do points A, K, D.

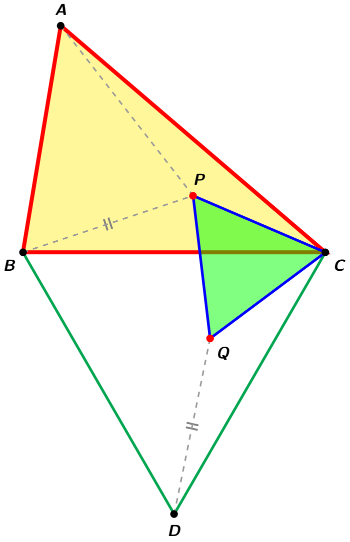
An optimization problem

A nice use of transformations comes in solving the following problem first studied by Fermat and Torricelli.

Problem

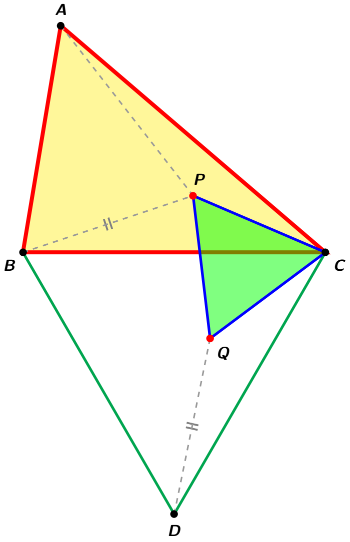
Given a triangle ABC, to find a point P in the plane of the triangle such that $PA + PB + PC$ has the least value possible.

We shall assume that no angle of the triangle exceeds 120°.



→ $\boldsymbol{P}$: candidate point.

→ Apply $\mathbf{R}_{C,60^\circ}$: $P \mapsto Q$, $B \mapsto D$.



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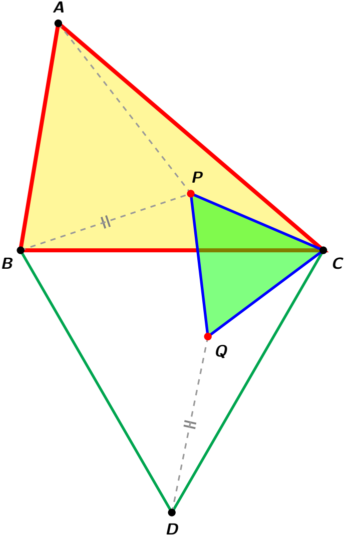
$\rightarrow$ $\triangle CPQ$, $\triangle BDC$: equilateral



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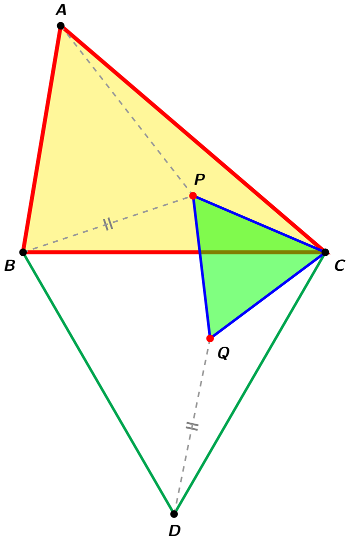
- $\rightarrow$ $\triangle CPQ$, $\triangle BDC$: equilateral
- $\rightarrow$ $PC = PQ$; $PB = QD$



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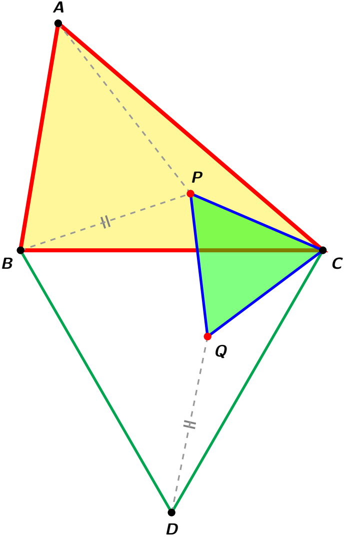
- $\rightarrow$ $\triangle CPQ$, $\triangle BDC$: equilateral
- $\rightarrow$ $PC = PQ$; $PB = QD$
- $\rightarrow$ $PA + PB + PC = DQ + QP + PA$



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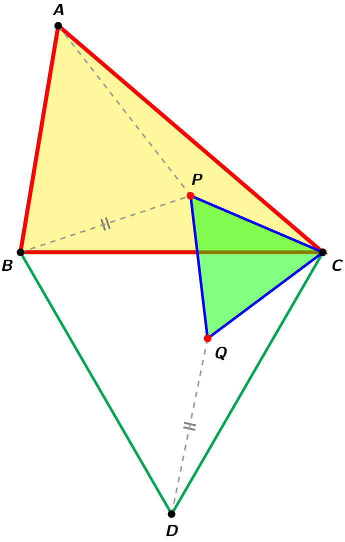
- $\rightarrow$ $PA + PB + PC \geq DA$



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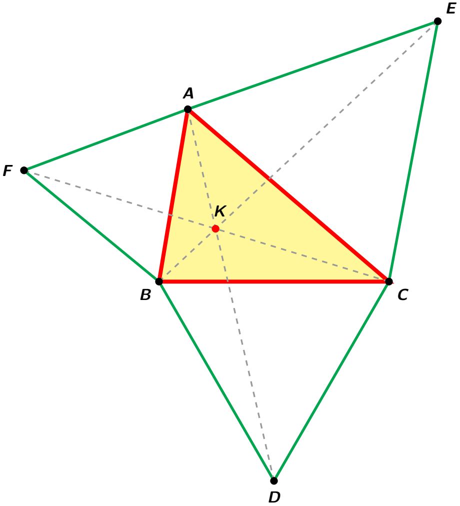
$\rightarrow$ $PA + PB + PC = DQ + QP + PA$

$\rightarrow$ $PA + PB + PC \geq DA$

$\rightarrow$ For equality: $\angle APC$, $\angle BPC$, $\angle APB$ all 120°. These are the conditions for P to be optimal.

Fermat point of a triangle

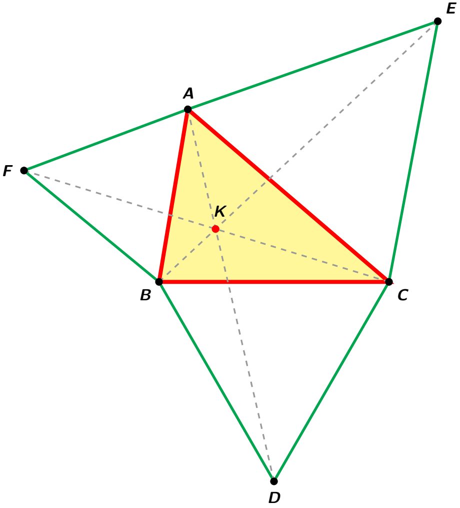


→ $\triangle BDC$, $\triangle CEA$, $\triangle AFB$: equilateral

→ ***AD***, ***BE***, ***CF*** have equal length, and they meet in the *Fermat point*, ***K***

→ ***AD***, ***BE***, ***CF*** make equal angles with one another

Fermat point of a triangle



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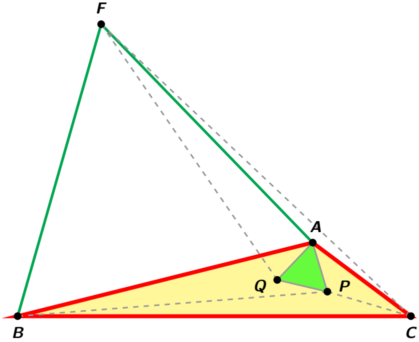
→ What happens if $\angle A > 120°$?

What happens if $\angle A > 120^\circ$



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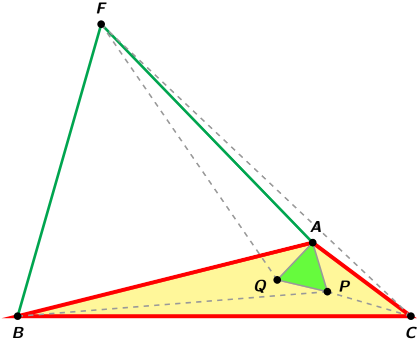
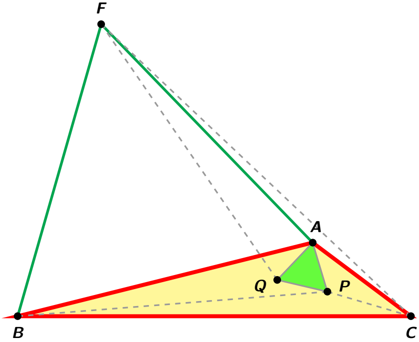
What happens if $\angle A > 120^\circ$



$\rightarrow$ $\boldsymbol{P}$: candidate point

$\rightarrow$ Apply $\mathbf{R}_{\boldsymbol{A},-60^\circ}$. It maps $\boldsymbol{P}$ to $\boldsymbol{Q}$ & $\boldsymbol{B}$ to $\boldsymbol{F}$. Crucial: segment $\boldsymbol{CF}$ lies 'outside' the figure.

What happens if $\angle A > 120^\circ$



- $\rightarrow$ P: candidate point
- $\rightarrow$ Apply $\mathbf{R}_{A,-60^\circ}$. It maps P to Q & B to F. Crucial: segment CF lies 'outside' the figure.
- $\rightarrow$ $PA + PB + PC$ is equal to $CP + PQ + QF$

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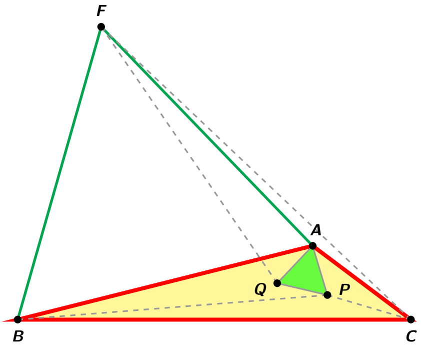


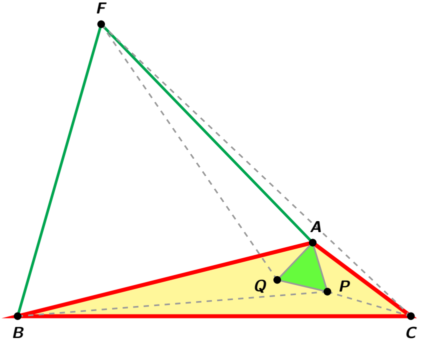
→ P: candidate point

→ Apply $\mathbf{R}_{A,-60^\circ}$. It maps P to Q & B to F. Crucial: segment CF lies 'outside' the figure.

→ $PA + PB + PC$ is equal to $CP + PQ + QF$

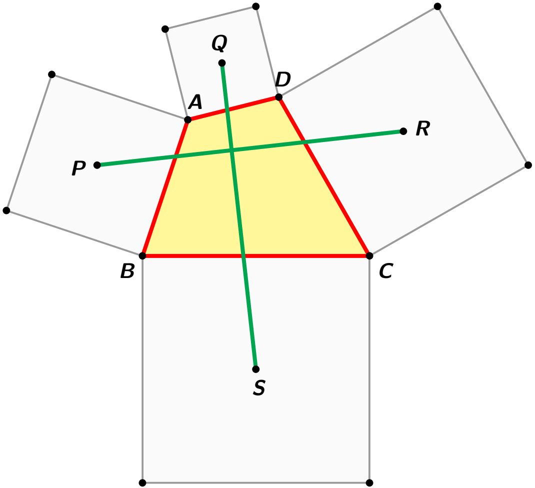
→ $CP + PQ + QF \geq CA + AF$, so $d(P) \geq d(A)$.

What happens if $\angle A > 120^\circ$



- $\rightarrow$ P: candidate point
- $\rightarrow$ Apply $\mathbf{R}_{A,-60^\circ}$. It maps P to Q & B to F. Crucial: segment CF lies 'outside' the figure.
- $\rightarrow$ $PA + PB + PC$ is equal to $CP + PQ + QF$
- $\rightarrow$ $CP + PQ + QF \geq CA + AF$, so $d(P) \geq d(A)$. Hence A is the optimizing point.

Von Aubel's quadrilateral theorem

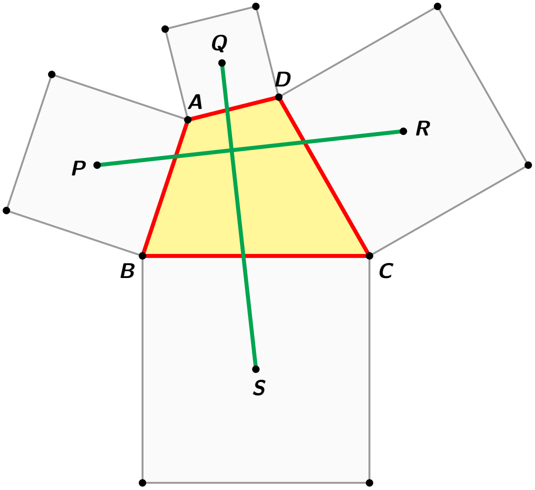


→ Quadrilateral ***ABCD***

→ Squares on its sides

→ Centres ***P***, ***Q***, ***R***, ***S***

Von Aubel's quadrilateral theorem

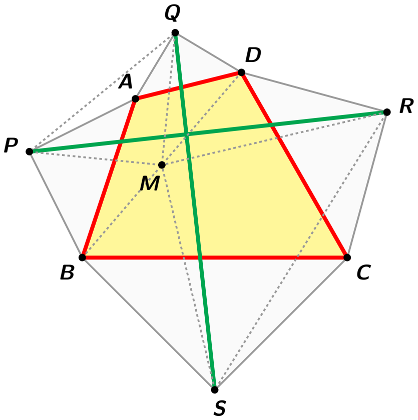


→ Quadrilateral ***ABCD***

→ Squares on its sides

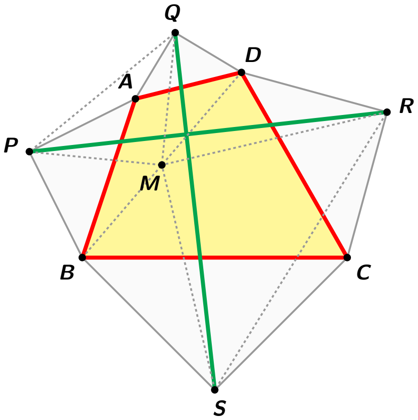
→ Centres ***P***, ***Q***, ***R***, ***S***

→ $PR = QS$, $PR \perp QS$



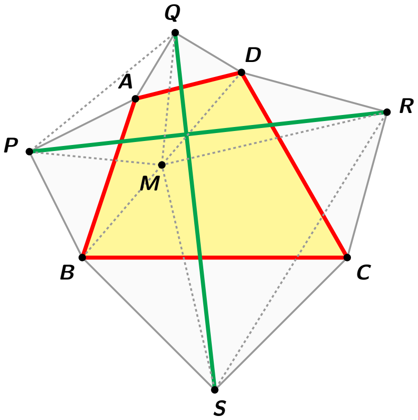
→ Apply $f = \mathbf{R}_{P,90^\circ}$, $g = \mathbf{R}_{Q,90^\circ}$.

$g \circ f$ is a half-turn.

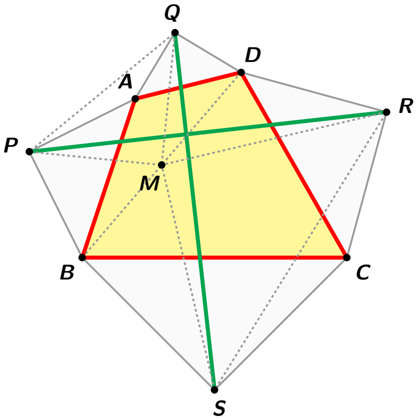


→ Apply $f = \mathrm{R}_{P,90^\circ}$, $g = \mathrm{R}_{Q,90^\circ}$.
$g \circ f$ is a half-turn.

→ $g \circ f(B) = g(A) = D$; so the centre of $g \circ f$ is the midpoint M of BD.



- → Apply $f = R_{P,90^\circ}$, $g = R_{Q,90^\circ}$. $g \circ f$ is a half-turn.
- → $g \circ f(B) = g(A) = D$; so the centre of $g \circ f$ is the midpoint M of BD.
- → $\triangle PMQ$ is isosceles right-angled at M. Same is true of $\triangle RMS$.



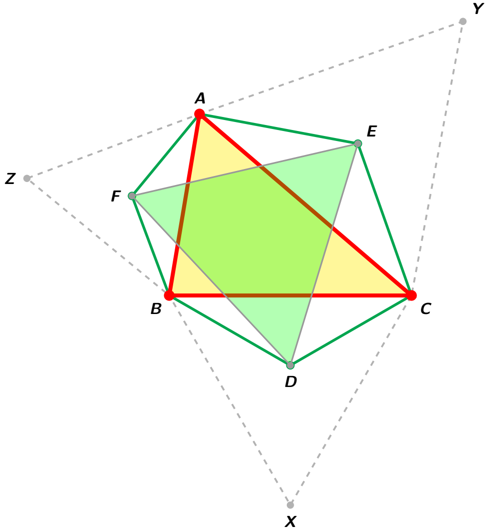
→ Apply $f = \mathrm{R}_{P,90^\circ}$, $g = \mathrm{R}_{Q,90^\circ}$. $g \circ f$ is a half-turn.

→ $g \circ f(B) = g(A) = D$; so the centre of $g \circ f$ is the midpoint M of BD.

→ $\triangle PMQ$ is isosceles right-angled at M. Same is true of $\triangle RMS$.

→ Now apply $h = \mathrm{R}_{M,90^\circ}$. It maps Q to P, S to R. Hence it maps QS to PR. Hence etc. □

Napoleon's theorem



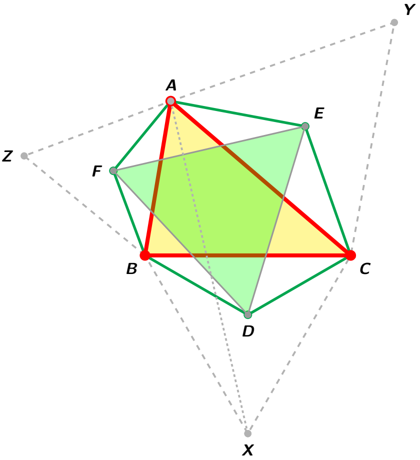
→ $\triangle ABC$: arbitrary

→ $\triangle BXC$, $\triangle CYA$, $\triangle AZB$: equilateral

→ D, E, F: their centroids (respectively); then:

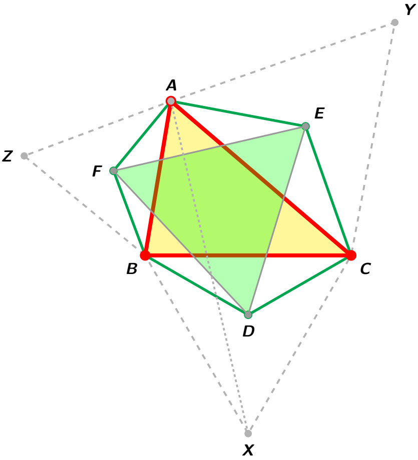
→ $\triangle DEF$ is equilateral

Napoleon's theorem: proof



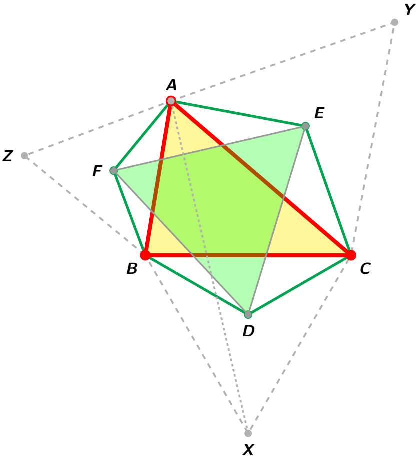
$\longrightarrow$ Apply $\mathbf{R}_{B,30^\circ}$ to $\triangle \boldsymbol{ABX}$, and then $\mathbf{E}_{B,1/\sqrt{3}}$

Napoleon's theorem: proof



- $\rightarrow$ Apply $\mathbf{R}_{B,30^\circ}$ to $\triangle \boldsymbol{ABX}$, and then $\mathbf{E}_{B,1/\sqrt{3}}$
- $\rightarrow$ $\triangle \boldsymbol{ABX}$ gets mapped to $\triangle \boldsymbol{FBD}$ so:

Napoleon's theorem: proof

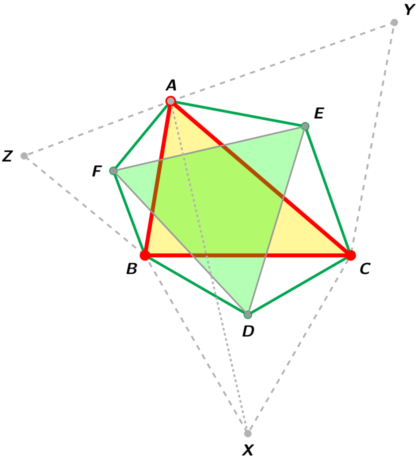


$\rightarrow$ Apply $\mathbf{R}_{B,30^\circ}$ to $\triangle ABX$, and then $\mathbf{E}_{B,1/\sqrt{3}}$

$\rightarrow$ $\triangle ABX$ gets mapped to $\triangle FBD$ so:

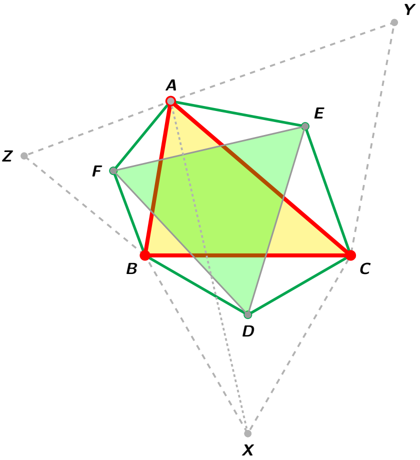
$\rightarrow$ $DF = AX/\sqrt{3}$, $\angle(DF, AX) = 30^\circ$

Napoleon's theorem: proof



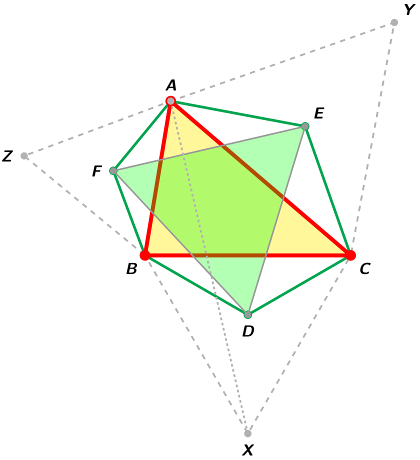
- $\rightarrow$ Apply $\mathbf{R}_{B,30^\circ}$ to $\triangle ABX$, and then $\mathbf{E}_{B,1/\sqrt{3}}$
- $\rightarrow$ $\triangle ABX$ gets mapped to $\triangle FBD$ so:
- $\rightarrow$ $DF = AX/\sqrt{3}$, $\angle(DF, AX) = 30^\circ$
- $\rightarrow$ Apply $\mathbf{R}_{C,-30^\circ}$ to $\triangle ACX$, then $\mathbf{E}_{C,1/\sqrt{3}}$.

Napoleon's theorem: proof



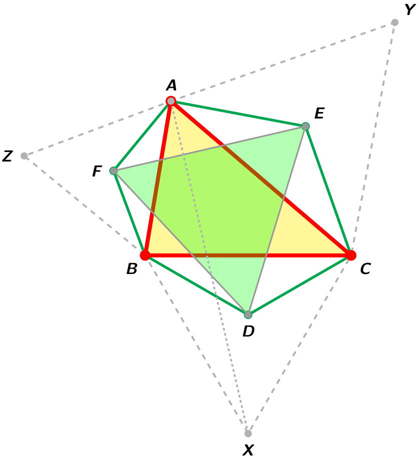
- $\rightarrow$ Apply $\mathbf{R}_{B,30^\circ}$ to $\triangle \boldsymbol{ABX}$, and then $\mathbf{E}_{B,1/\sqrt{3}}$
- $\rightarrow$ $\triangle \boldsymbol{ABX}$ gets mapped to $\triangle \boldsymbol{FBD}$ so:
- $\rightarrow$ $\boldsymbol{DF = AX/\sqrt{3}}$, $\boldsymbol{\angle(DF, AX) = 30^\circ}$
- $\rightarrow$ Apply $\mathbf{R}_{C,-30^\circ}$ to $\triangle \boldsymbol{ACX}$, then $\mathbf{E}_{C,1/\sqrt{3}}$. We get: $\boldsymbol{DE = AX/\sqrt{3}}$ and $\angle\boldsymbol{(DE, AX) = -30^\circ}$. So:

Napoleon's theorem: proof



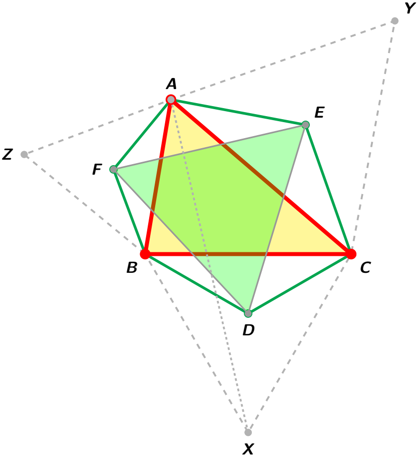
- → Apply $\mathbf{R}_{B,30^\circ}$ to $\triangle ABX$, and then $\mathbf{E}_{B,1/\sqrt{3}}$
- → $\triangle ABX$ gets mapped to $\triangle FBD$ so:
- → $DF = AX/\sqrt{3}$, $\angle(DF, AX) = 30^\circ$
- → Apply $\mathbf{R}_{C,-30^\circ}$ to $\triangle ACX$, then $\mathbf{E}_{C,1/\sqrt{3}}$. We get: $DE = AX/\sqrt{3}$ and $\angle(DE, AX) = -30^\circ$. So:
- → $DE = DF$, $\angle(DF, DE) = 60^\circ$

Napoleon's theorem: proof



- $\rightarrow$ Apply $\mathbf{R}_{B,30^\circ}$ to $\triangle ABX$, and then $\mathbf{E}_{B,1/\sqrt{3}}$
- $\rightarrow$ $\triangle ABX$ gets mapped to $\triangle FBD$ so:
- $\rightarrow$ $DF = AX/\sqrt{3}$, $\angle(DF, AX) = 30^\circ$
- $\rightarrow$ Apply $\mathbf{R}_{C,-30^\circ}$ to $\triangle ACX$, then $\mathbf{E}_{C,1/\sqrt{3}}$. We get: $DE = AX/\sqrt{3}$ and $\angle(DE, AX) = -30^\circ$. So:
- $\rightarrow$ $DE = DF$, $\angle(DF, DE) = 60^\circ$
- $\rightarrow$ Hence $\triangle DEF$ is equilateral

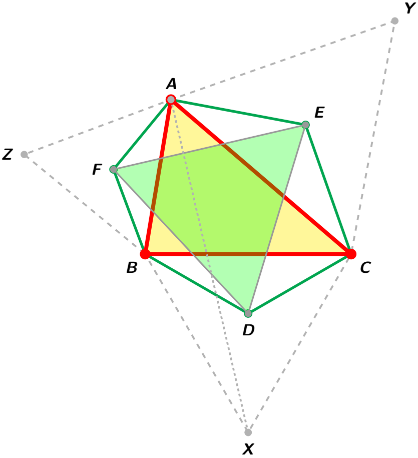
Napoleon's theorem: second proof



$\rightarrow$ $\angle BFA = \angle AEC = \angle CDB = 120°$,

and $3 \times 120° = 360°$

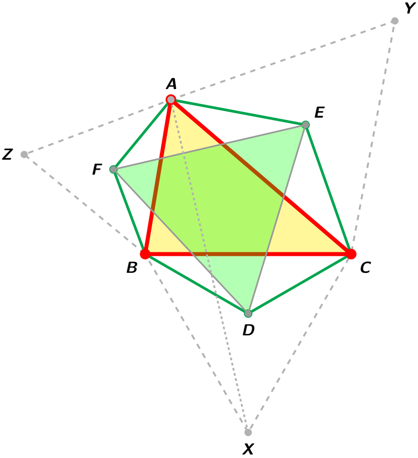
Napoleon's theorem: second proof



→ $\angle BFA = \angle AEC = \angle CDB = 120°$, and $3 \times 120° = 360°$

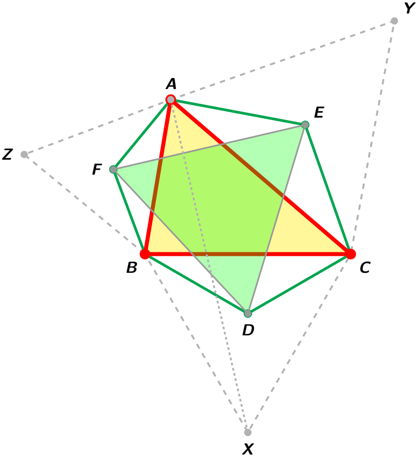
→ Let $f = R_{D,120°} \circ R_{E,120°} \circ R_{F,120°}$; f maps B to itself. Hence $f = Id$.

Napoleon's theorem: second proof



- $\rightarrow$ $\angle BFA = \angle AEC = \angle CDB = 120^\circ$, and $3 \times 120^\circ = 360^\circ$
- $\rightarrow$ Let $f = \mathbf{R}_{D,120^\circ} \circ \mathbf{R}_{E,120^\circ} \circ \mathbf{R}_{F,120^\circ}$; f maps B to itself. Hence $f = Id$.
- $\rightarrow$ Hence $\mathbf{R}_{E,120^\circ} \circ \mathbf{R}_{F,120^\circ} = \mathbf{R}_{D,240^\circ}$

Napoleon's theorem: second proof



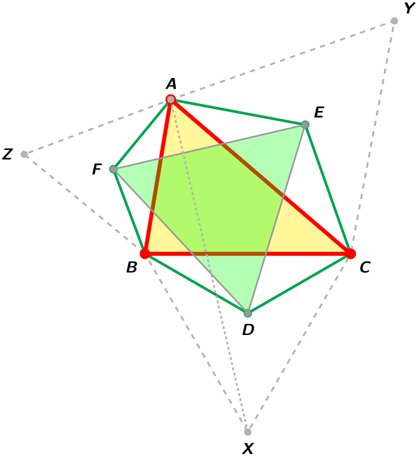
→ $\angle BFA = \angle AEC = \angle CDB = 120^\circ$, and $3 \times 120^\circ = 360^\circ$

→ Let $f = \mathbf{R}_{D,120^\circ} \circ \mathbf{R}_{E,120^\circ} \circ \mathbf{R}_{F,120^\circ}$; f maps B to itself. Hence $f = Id$.

→ Hence $\mathbf{R}_{E,120^\circ} \circ \mathbf{R}_{F,120^\circ} = \mathbf{R}_{D,240^\circ}$

→ Recalling the way rotation maps are composed, we see that $\angle DEF = 60^\circ = \angle DFE$.

Napoleon's theorem: second proof



→ $\angle BFA = \angle AEC = \angle CDB = 120^\circ$, and $3 \times 120^\circ = 360^\circ$

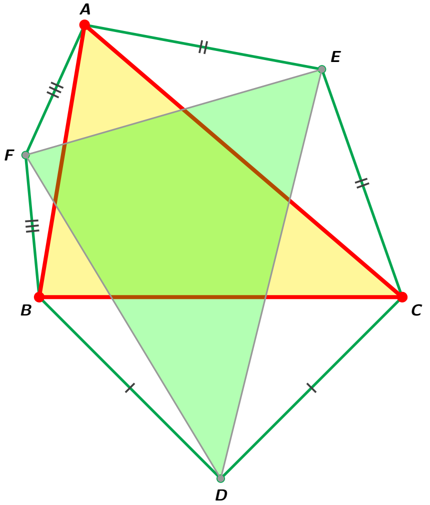
→ Let $f = \mathbf{R}_{D,120^\circ} \circ \mathbf{R}_{E,120^\circ} \circ \mathbf{R}_{F,120^\circ}$; f maps B to itself. Hence $f = Id$.

→ Hence $\mathbf{R}_{E,120^\circ} \circ \mathbf{R}_{F,120^\circ} = \mathbf{R}_{D,240^\circ}$

→ Recalling the way rotation maps are composed, we see that $\angle DEF = 60^\circ = \angle DFE$.

→ Hence $\triangle DEF$ is equilateral

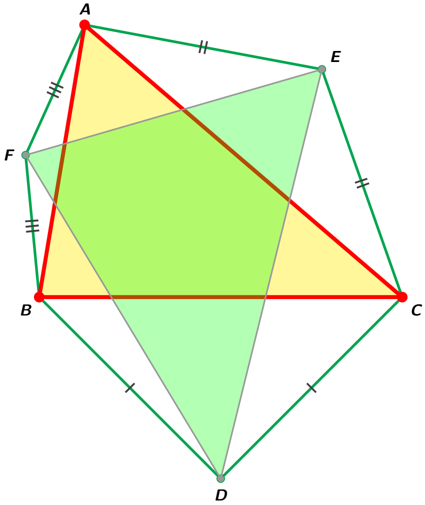
Napoleon's theorem: generalized version



→ $\triangle ABC$: arbitrary

→ $\triangle BDC$, $\triangle CEA$, $\triangle AFB$: all isosceles, with apex angles α, β, γ (resp), $\alpha + \beta + \gamma = 360°$; then:

Napoleon's theorem: generalized version

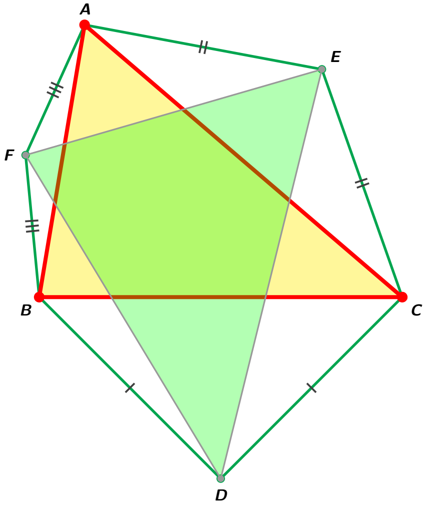


→ $\triangle \boldsymbol{ABC}$: arbitrary

→ $\triangle \boldsymbol{BDC}$, $\triangle \boldsymbol{CEA}$, $\triangle \boldsymbol{AFB}$: all isosceles, with apex angles α, β, γ (resp), $\alpha + \beta + \gamma = 360°$; then:

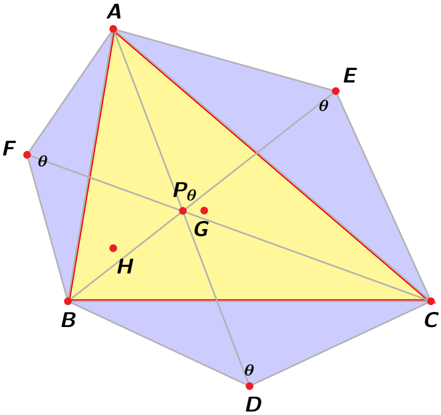
→ $\triangle \boldsymbol{DEF}$ has angles $\alpha/2$, $\beta/2$, $\gamma/2$ (resp)

Napoleon's theorem: generalized version



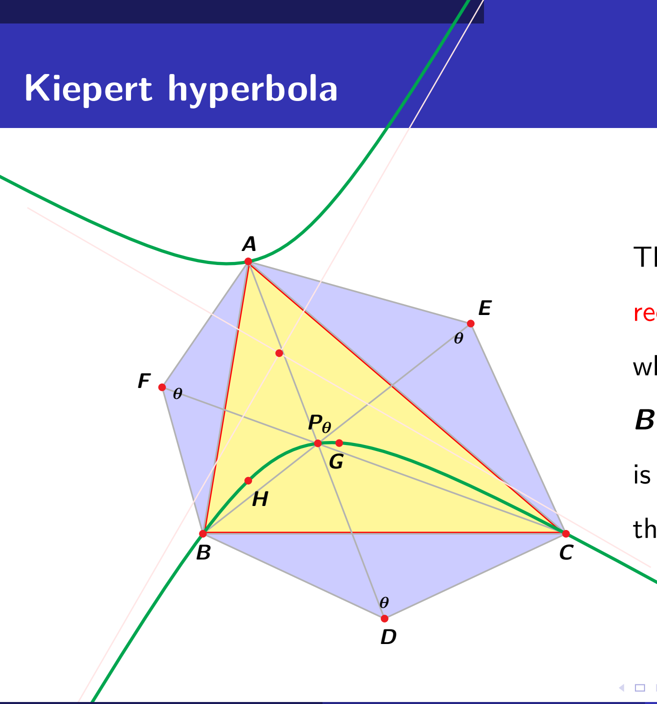
- → $\triangle ABC$: arbitrary
- → $\triangle BDC$, $\triangle CEA$, $\triangle AFB$: all isosceles, with apex angles α, β, γ (resp), $\alpha + \beta + \gamma = 360°$; then:
- → $\triangle DEF$ has angles $\alpha/2$, $\beta/2$, $\gamma/2$ (resp)
- → Proof: Immediate . . .

Napoleon tweaked ...



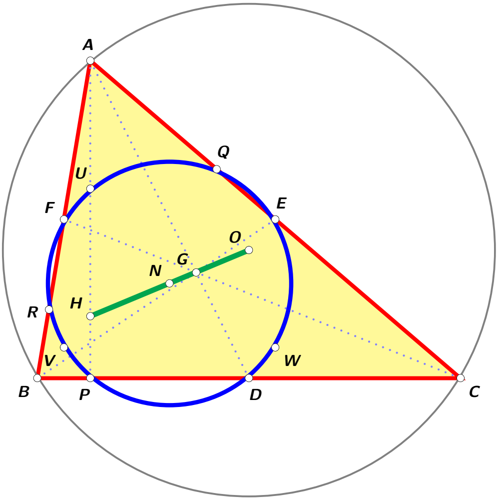
- → $\triangle ABC$: arbitrary
- → $\triangle BDC$, $\triangle CEA$, $\triangle AFB$: similar isosceles, all with apex angle θ
- → AD, BE, CF concur at $P = P_\theta$
- → Locus of P_θ?

Kiepert hyperbola



The locus of $\boldsymbol{P_\theta}$ is a rectangular hyperbola which passes through $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{H}$ and $\boldsymbol{G}$ (here $\boldsymbol{H}$ is the orthocentre and $\boldsymbol{G}$ the centroid of $\triangle \boldsymbol{ABC}$)!

Nine-point circle



→ $\triangle ABC$: arbitrary

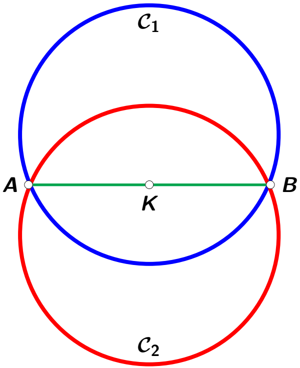
→ D, E, F: midpoints of sides

→ P, Q, R: feet of altitudes

→ U, V, W: midpoints of segments HA, HB, HC

→ Points D, E, F, P, Q, R, U, V, W all lie on a circle centred at the midpoint N of OH

Basic idea for the proof

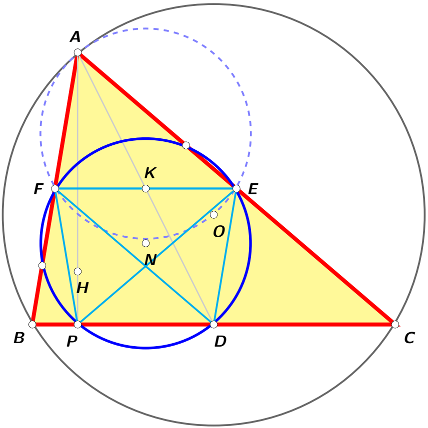


Given two circles $\mathcal{C}_1$, $\mathcal{C}_2$ of equal size, sharing a chord AB, there are at least two distinct isometric maps which map one circle to the other:

- → Reflection in line AB
- → Half-turn about the midpoint K of AB.

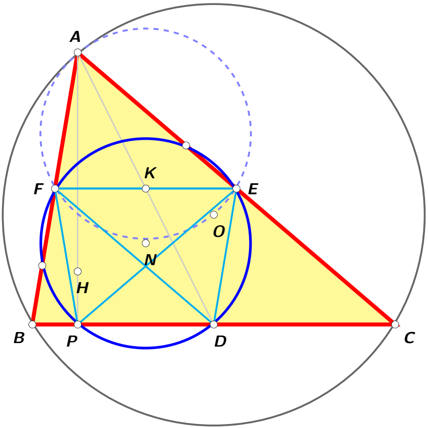
Notation: $\omega(ABC)$ denotes the circumcircle of $\triangle ABC$, etc

Nine-point circle: first part of proof



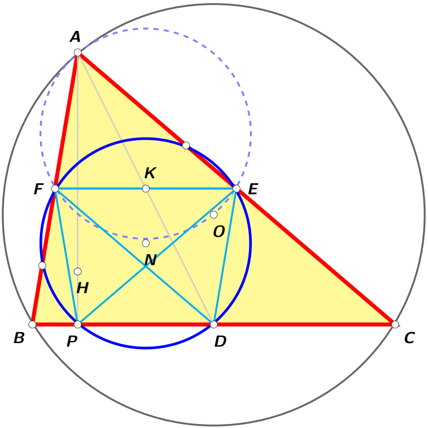
1. $\mathbf{E}_{A,1/2}$ maps $\omega(ABC)$ to $\omega(AFE)$

Nine-point circle: first part of proof



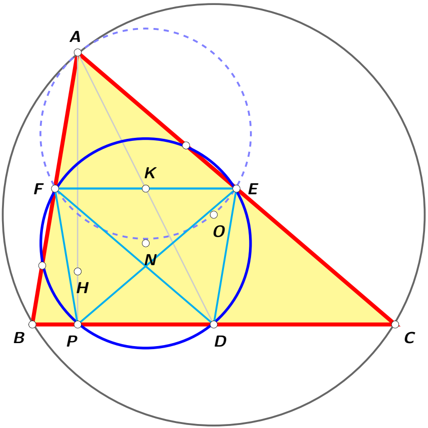
1. $\mathbf{E}_{A,1/2}$ maps $\omega(ABC)$ to $\omega(AFE)$
2. $\mathbf{M}_{EF}$ maps $\omega(AFE)$ to $\omega(PFE)$

Nine-point circle: first part of proof



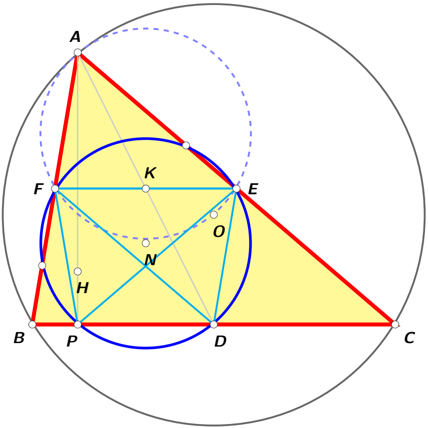
1. $\mathbf{E}_{A,1/2}$ maps $\omega(ABC)$ to $\omega(AFE)$
2. $\mathbf{M}_{EF}$ maps $\omega(AFE)$ to $\omega(PFE)$
3. $\mathbf{H}_{K}$ maps $\omega(AFE)$ to $\omega(DEF)$

Nine-point circle: first part of proof



1. $\mathbf{E}_{A,1/2}$ maps $\omega(ABC)$ to $\omega(AFE)$
2. $\mathbf{M}_{EF}$ maps $\omega(AFE)$ to $\omega(PFE)$
3. $\mathbf{H}_K$ maps $\omega(AFE)$ to $\omega(DEF)$
4. Hence P lies on $\omega(DEF)$. By symmetry, do so Q, R.

Nine-point circle: first part of proof



1. $\mathbf{E}_{A,1/2}$ maps $\omega(ABC)$ to $\omega(AFE)$
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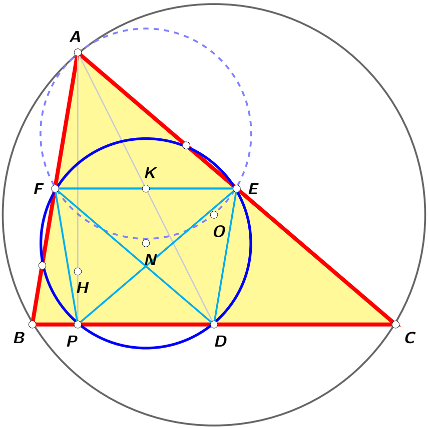
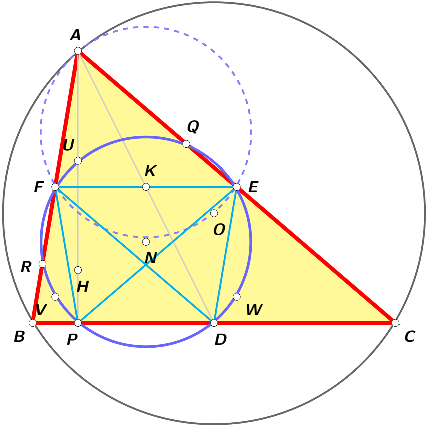
4. Hence P lies on $\omega(DEF)$. By symmetry, do so Q, R.
5. So $\omega(DEF) = \omega(PQR)$.

Nine-point circle: first part of proof



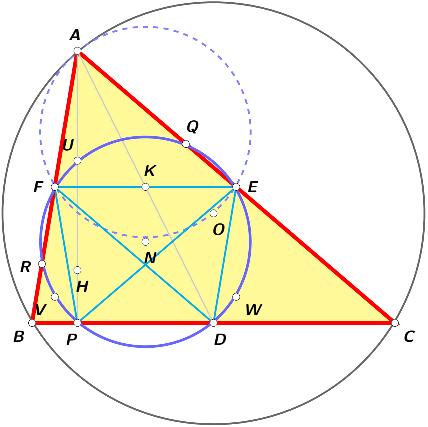
1. $\mathbf{E}_{A,1/2}$ maps $\omega(ABC)$ to $\omega(AFE)$
2. $\mathbf{M}_{EF}$ maps $\omega(AFE)$ to $\omega(PFE)$
3. $\mathbf{H}_{K}$ maps $\omega(AFE)$ to $\omega(DEF)$
4. Hence P lies on $\omega(DEF)$. By symmetry, do so Q, R.
5. So $\omega(DEF) = \omega(PQR)$.
6. Centre: $E + F - A/2$ which is $(A + B + C)/2 = H/2 = N$

Nine-point circle: second part of proof



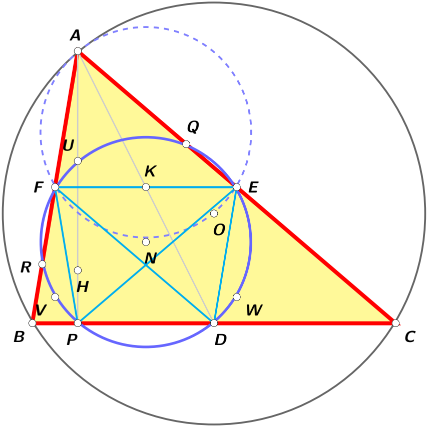
1. $\mathbf{E}_{H,1/2}$ maps $\omega(\boldsymbol{ABC})$ to $\omega(\boldsymbol{UVW})$

Nine-point circle: second part of proof

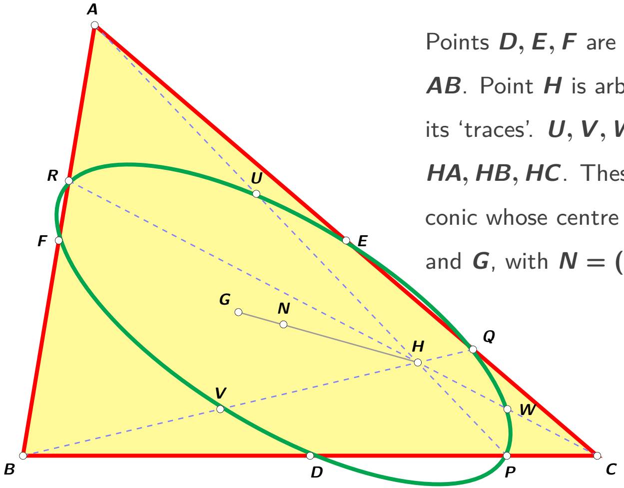


1. $\mathbf{E}_{H,1/2}$ maps $\omega(ABC)$ to $\omega(UVW)$
2. Hence the centre of $\omega(UVW)$ is $H/2 = N$, and its radius is half the radius of $\omega(ABC)$

Nine-point circle: second part of proof



1. $\mathbf{E}_{H,1/2}$ maps $\omega(\boldsymbol{ABC})$ to $\omega(\boldsymbol{UVW})$
2. Hence the centre of $\omega(\boldsymbol{UVW})$ is $\boldsymbol{H}/2 = \boldsymbol{N}$, and its radius is half the radius of $\omega(\boldsymbol{ABC})$
3. Hence $\omega(\boldsymbol{UVW})$ coincides with $\omega(\boldsymbol{DEF})$ and $\omega(\boldsymbol{PQR})$.



Points D, E, F are midpoints of BC, CA, AB. Point H is arbitrary, and P, Q, R are its 'traces'. U, V, W are the midpoints of HA, HB, HC. These nine points lie on a conic whose centre N is collinear with H and G, with $N = (3G + H)/4$. Well!

Part II

Frieze Patterns

Mosaic

Human beings have used mosaic as an art form for centuries. The Islamic cultures in particular have developed this art form to a very high level.

Mosaic

Human beings have used mosaic as an art form for centuries. The Islamic cultures in particular have developed this art form to a very high level.

Mosaic can be *one dimensional*, as in a frieze pattern, or *two dimensional*, as in wallpaper or floor tilings.

The most stunning exhibitions of tiling patterns are seen in the Alhambra Palace in Granada, Spain.

The seven frieze patterns

Frieze patterns are all around us (look around you and check this out).

It can be shown that the underlying symmetry group of a frieze pattern is one of just seven possibilities.

This may come as a surprise, but it can be proved. The individual details of the motifs used in the pattern may differ, but the *symmetry groups* are just seven in number.

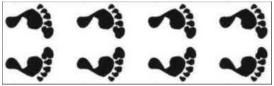
We take a quick look at these seven possibilities.

Translations only

Frieze pattern type T: . . . F F F F F F F . . .

Translations and horizontal axis reflection

Frieze pattern type *TX*: . . . B B B B B B B . . .

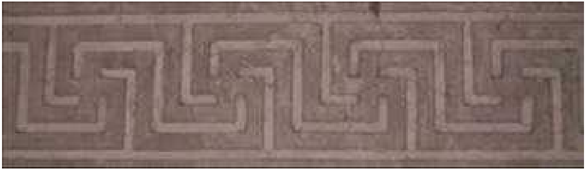


Translations and vertical axis reflection

Frieze pattern type *TY*: ... A A A A A A A ...

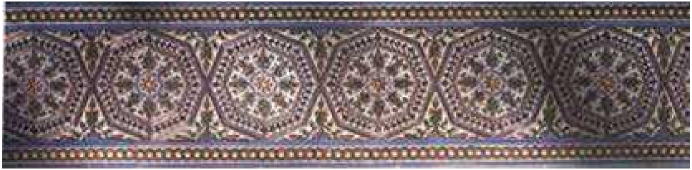
Translations and half turns

Frieze pattern type *TH*: . . . N N N N N N N . . .



Translations and horizontal and vertical reflections

Frieze pattern type *THXY*: . . . X X X X X X X . . .



Translations and glide reflections

Frieze pattern type *TG*: ... b p b p b p b p ...

Translations and vertical reflections + glides

Frieze pattern type *TGY*: ... p q b d p q b d p q b d p q ...

- Max Jeger, *Transformation Geometry* (1966)
- E A Maxwell, *Geometry By Transformations* (Cambridge Univ Press, SMP)
- I M Yaglom, *Geometric Transformations I* and *Geometric Transformations II* (MAA)