# The Topos of Finite Sets and Algebraic Geometry

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- The talk could also be given the title *Scheme Theory for Discrete Mathematicians*.
- Hilbert is supposed to have had an interest in an "elementary" approach to algebraic geometry. Here elementary is in the sense of logic and not number theory.

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3. Given maps $p : A \to C$ and $q : B \to C$ we can form the *fibred* product $A \times_C B$ consisting consisting of pairs $(a, b)$ where $p(a) = q(b)$. This too is a finite set.
4. We have a special set $\Omega$ (doublet) and an inclusion $1 \to \Omega$ so that each subset $S$ of a set $A$ is of the form $A \times_\Omega 1$ for a suitable map $s : A \to \Omega$. It follows that the power set $P(A) = \hom(A, \Omega)$ of a finite set is also a finite set.

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5. Composition of morphisms is associative.

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2. Given morphisms $p : A \to C$ and $q : B \to C$, there is an object $A \times_C B$ with morphisms $\pi_A : A \times_C B \to A$ and $\pi_B : A \times_C B \to B$ so that $p \circ \pi_A = q \circ \pi_B$. Moreover, given an object $T$ and morphisms $f : T \to A$ and $g : T \to B$ so that $p \circ f = q \circ g$, there is a *unique* morphism $(f, g) : T \to A \times_C B$ so that $\pi_A \circ (f, g) = f$ and $\pi_B \circ (f, g) = g$.

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These two are equivalent to the assertion that *finite limits exist* in the sense of category theory.

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This has the property that given a relation $R \subset A \times B$ (here $\subset$ indicates a monomorphism) between $B$ and $A$ there is a unique morphism $c_R : B \to P(A)$ so that $R$ is canonically isomorphic to $S_A \times_{A \times P(A)} (A \times B)$.

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- The object $\hom(A, B)$ classifying morphisms $A \to B$.

Note that **0** is *deduced* unlike traditional set theory!

# Monoid and Groups

A monoid object in a topos is $(M, e, m)$ where $e : \mathbf{1} \to M$ is the identity "element" and $m : M \times M \to M$ is the multiplication. The usual axioms for a monoid can be written as follows:

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$$m \circ (e \circ s_A, 1_M) = m \circ (1_M, e \circ s_A) = \quad 1_M$$
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In the topos of finite sets, this gives the notion of a finite group.

# Rings and Subcategories

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This gives us the categories $\mathcal{M}$ of finite monoids, $\mathcal{G}$ of finite groups, $\mathcal{R}$ of finite rings and $\mathcal{C}$ of commutative finite rings.

# Functors

A *functor* $F$ from a category $\mathcal{C}$ to another category $\mathcal{D}$ is an assignment of an object $F(A)$ in the second category to an object $A$ in the first category and a morphism $F(f) : F(A) \to F(B)$ to a morphism $f : A \to B$ in the first category.

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# Affine Schemes

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We can now define the affine scheme $V(X_1, \ldots, X_p; f_1, \ldots, f_q)$ as

$$V(X_1, \ldots, X_p; f_1, \ldots, f_q)(A) = \{(a_1, \ldots, a_p) | f_i(a_1, \ldots, a_p) = 0, \forall i\}$$

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Further, note that the product of affine schemes is also an affine scheme

$$\begin{aligned} V(X_1, \ldots, X_p; f_1, \ldots, f_q) \times V(Y_1, \ldots, Y_k; g_1, \ldots, g_l) = \\ V(X_1, \ldots, X_p, Y_1, \ldots, Y_k; f_1, \ldots, f_q, g_1, \ldots, g_l) \end{aligned}$$

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In order to validate this definition we need to compare it with the usual one.

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An affine group scheme is a functor from $\mathcal{C}$ to finite groups such that the underlying set-valued functor is an affine scheme.
For example, we can send each finite ring to the underlying additive group which gives the additive group $\mathbb{G}_a$.
A *vector group scheme* over $S$ is the fibre product of:

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The importance of vector group schemes over $S$ is that it is the *dual* category of the "usual" category $\mathrm{C}oh(S)$ of coherent sheaves on $S$.

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It remains to be seen whether all or most proofs in algebraic geometry can be achieved within the topos of finite sets!

# Justification

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This proof is much to long to present here and also goes outside the topos of finite sets!

# More Justification

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What happens if we replace $\mathcal{C}$ by $\mathcal{R}$?

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What happens if we replace $\mathcal{C}$ by $\mathcal{R}$? This appears to be a possible approach to talking about non-commutative algebraic geometry.

Thank you for your attention