A talk based on

The Hodge Conjecture for general Prym varieties

by

Indranil Biswas

School of Maths, TIFR, Homi Bhabha Road, Mumbai 400 005.

indranil@math.tifr.res.in

and

Kapil H. Paranjape

IMSc, CIT Campus, Tharamani, Chennai 600 113.

kapil@imsc.res.in

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As this is a Millennium Prize Problem there are many ways to find out.

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So what is an Abelian Variety?

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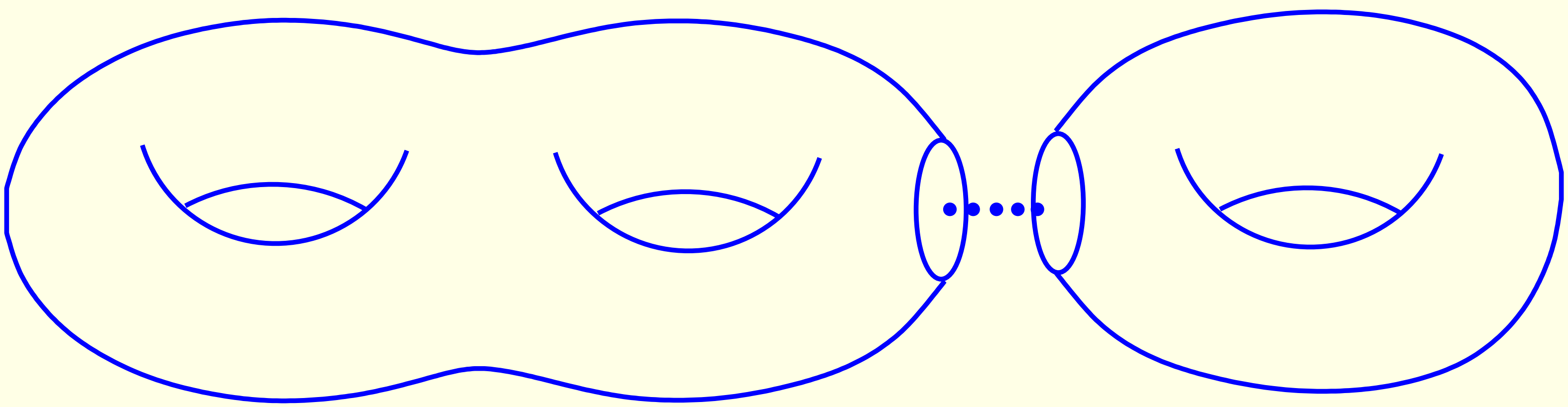

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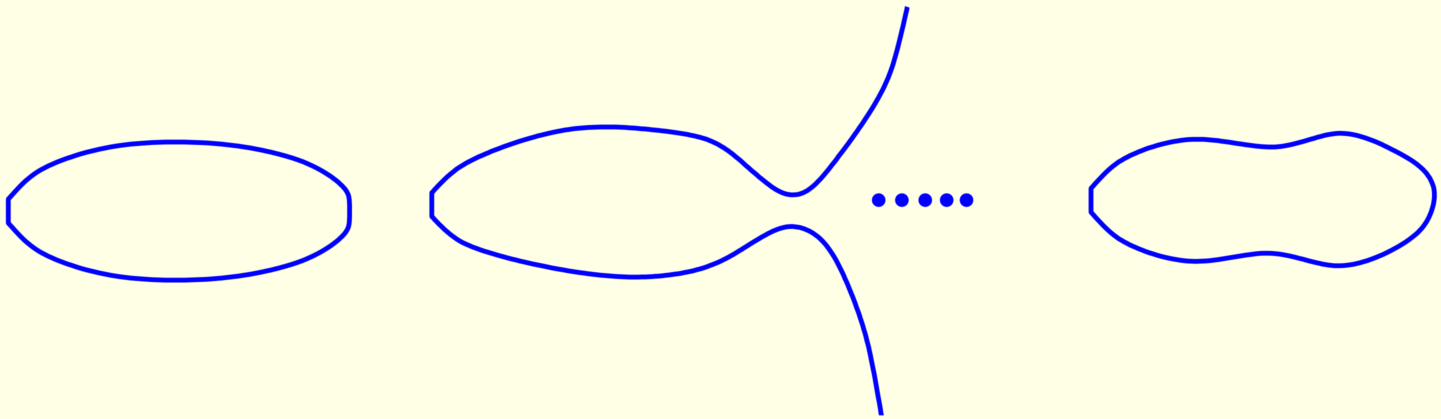
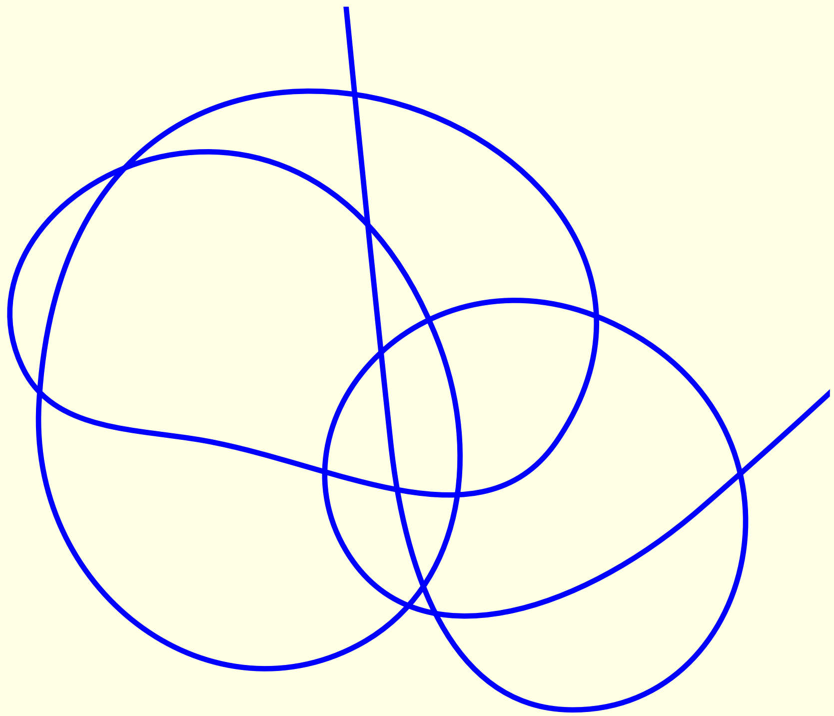
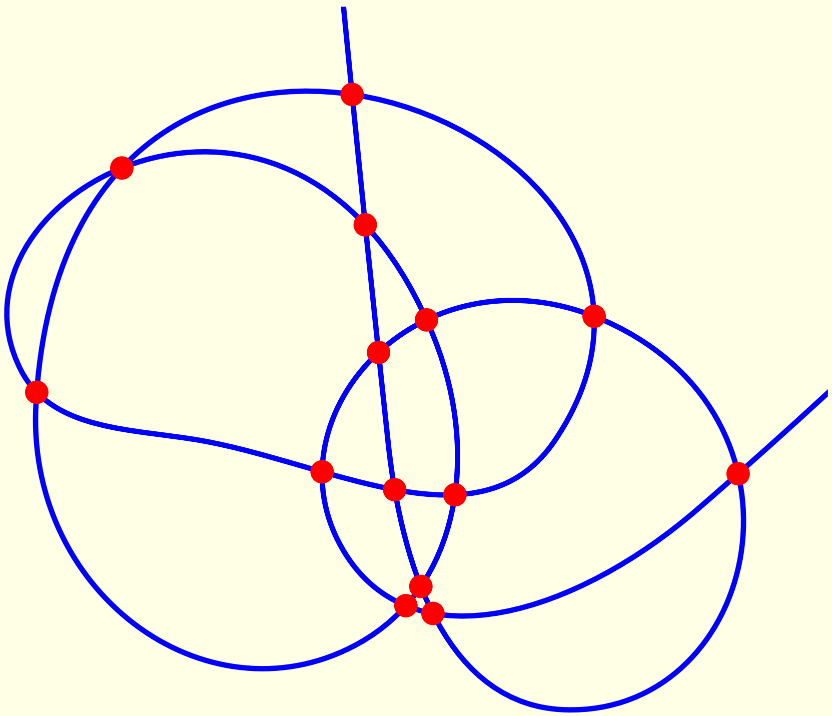
Start with a curve X, by which we mean,

or a projective algebraic curve





which we can also think of as a plane curve



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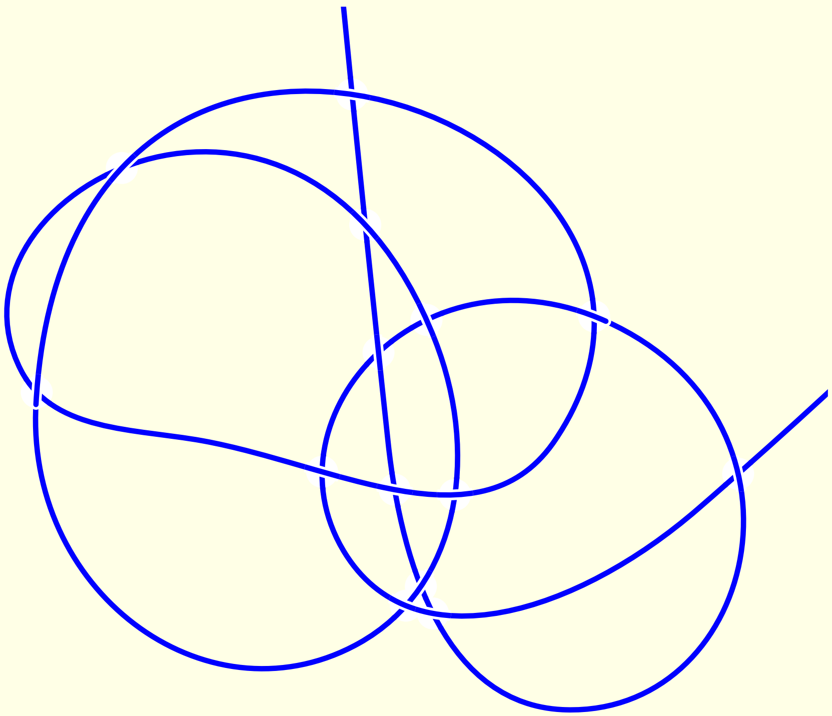

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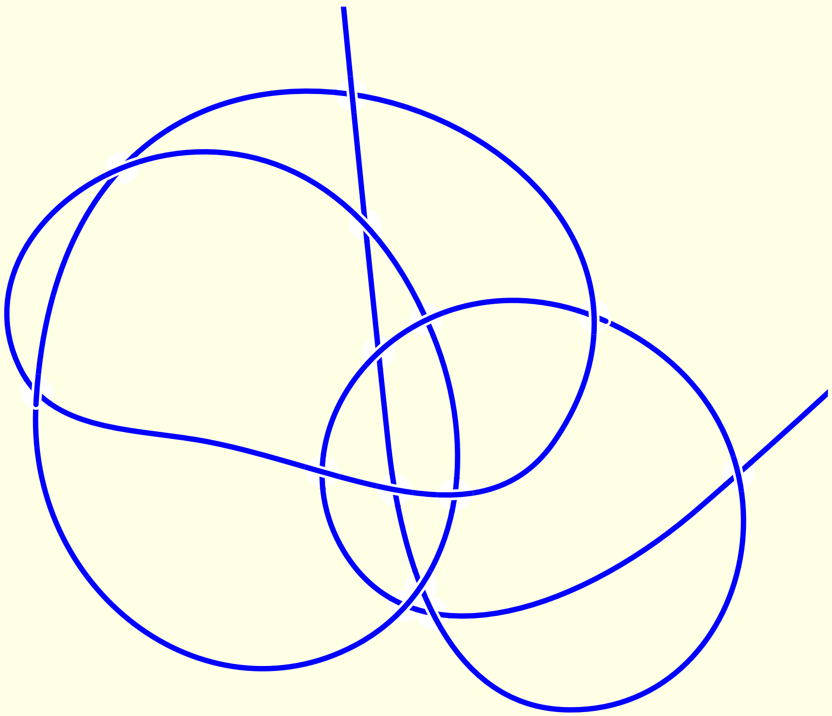

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Which can be done in many ways.

The r-tuples of points on a curve X form a variety

$$S^r(X) = X \times X \times \cdots \times X / \text{ symmetric group action}$$

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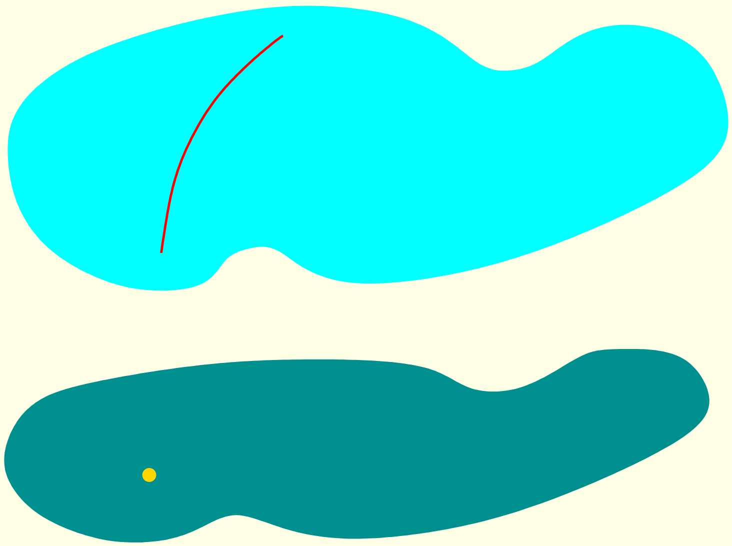

***Warning**: Curves in algebraic geometry are parametric only when the

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The Jacobian variety can then be obtained as the quotient

$$J(X) = S^r(X)/ \text{ equivalence relation}$$

The group structure is obtained by concatenating two r-tuples to get a $2r$-tuple which is then reduced to an r-tuple via equivalence.

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Fact: Every Abelian variety A occurs as a connected subgroup of $J(X)$ for a suitable curve X.

If $f : X \to Y$ is a map of curves we get an induced map $f_* : J(X) \to J(Y)$.

When the map f of curves is 2-to-1 the (connected part of the) kernel of f_* is called a Prym variety.

The Hodge Conjecture for an Abelian Variety A can be re-stated group-theoretically. Specifically, one defines the Mumford-Tate group $G(A)$, which occurs as a subgroup of the symplectic group*.

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* The symplectic group consists of $2g \times 2g$ matrices M such that

$$M^t \cdot \begin{pmatrix} 0 & \dots & 0 & -1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & -1 \\ 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix} \cdot M = \begin{pmatrix} 0 & \dots & 0 & -1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & -1 \\ 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix}$$

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The Hodge Conjecture for Abelian varieties can thus be studied class by class.

Let L be the list of those subgroups $G(A)$ of the symplectic group which are associated with an A for which the Hodge conjecture is known.

There are two ways of proving the Hodge conjecture for a given Abelian variety A:

- **Easy Way**

- **Hard Way**

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- **Easy Way** Show that $G(A)$ is in L.

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. . . and get a Fields' Medal if you add infinitely many elements.

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As a start we must understand the list L.

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So Prym varieties should be special, or our result would be trivial.

On the other hand we are proving the result for the general Prym variety.

. . . sounds confusing!

Let's de(con)fuse the situation.

An Abelian variety is (the locus of zeroes of) a system of algebraic equations in many variables like*

$$\begin{array}{cccccccccccc} 24/6 & x_0^4 & + & 33 & x_1^5 & + & \sqrt{131} & x_2^5 & + & \zeta(3) & x_3^5 & = & 0 \\ 34/57 & x_0x_1^4 & + & e & x_1x_2^4 & + & 14.1 & x_2x_3^4 & + & 117^{1/7} & x_3x_4^4 & = & 0 \\ \pi & x_5^2x_0^3 & + & e^{2\pi\sqrt{-1}/5} & x_1^2x_2^3 & + & \gamma & x_2^5x_3 & + & 102 & x_3^2x_4^3 & = & 0 \end{array}$$

The coefficients of these equations could be:

- constants i. e. rational or algebraic numbers. Or,

- transcendental.

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A more general set of equations

$$\begin{array}{cccccccccccc} a_1 & x_0^4 & + & a_2 & x_1^5 & + & a_3 & x_2^5 & + & a_4 & x_3^5 & = & 0 \\ b_1 & x_0 x_1^4 & + & b_2 & x_1 x_2^4 & + & b_3 & x_2 x_3^4 & + & b_4 & x_3 x_4^4 & = & 0 \\ c_1 & x_5^2 x_0^3 & + & c_2 & x_1^2 x_2^3 & + & c_3 & x_2^5 x_3 & + & c_4 & x_3^2 x_4^3 & = & 0 \end{array}$$

is one where the coefficients in the equations above are replaced by new variables; call these new variables parameters.

We then obtain family of varieties depending on a set of parameters (or moduli). Each actual variety is obtained by substituting values for the parameters—this is called specialising.

For example, the equation $y^2 = x^3 - 1$ is a specialisation of the equation $y^2 = x^3 + ax + b$ obtained by putting $a = 0$ and $b = -1$.

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So we can specialise the equation $y^2 = x^3 + \pi x + e$ by putting any values in place of π and e that we like*.

*We can even be perverse and replace π by $\frac{22}{7}$—or 3!

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The condition that the equations define an Abelian variety can be written as a system of equations

$$\begin{aligned} f_1(a_1, \ldots, c_4) &= 0 \\ &\ldots \\ f_p(a_1, \ldots, c_4) &= 0 \end{aligned}$$

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in the parameters. Let us assume that a minimal such system of equations has been found.

An Abelian variety A is general if the actual coefficients of the equations defining A satisfy no equations other these.

An Abelian variety A is special otherwise.

We can write down an additional collection of conditions

$$
\begin{aligned}
g_1(a_1, \ldots, c_4) &= 0 \\
&\cdots \\
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that the coefficients of the equations need to satisfy in order that the Abelian variety be a Prym variety.

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Hence we have explained the result.