Pierre Deligne: The Person and The Work

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1 About this talk

The occasion for this talk is the announcement of the award of the Abel Prize for 2013 to Professor (Emeritus) Pierre Deligne of the Institute for Advanced Study, Princeton, USA. It was suggested to me that one should talk about the Abel Prize as the talk was intended for a general audience. That would be proper since the work of the Abel committee does give us the opportunity to celebrate mathematicians and mathematics — we thank them for that. However, at least in this particular case, the man is far greater than the award. So I re-titled this talk to "Pierre René, Viscount Deligne: The Person and The Work".

1.1 Sources

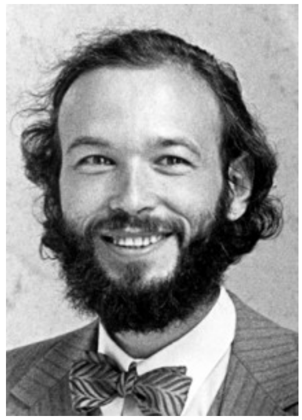
This talk has been prepared using the following sources:

1. Wikipedia. `http://en.wikipedia.org/wiki/Deligne`
2. St. Andrews, Mathematical Biographies. `http://www-history.mcs.st-andrews.ac.uk/Biographies/Deligne.html`
3. Simons Foundation, Biography and interview of Deligne. `https://simonsfoundation.org/science_lives_video/pierre-deligne/`
4. Faculty Page at Princeton. `http://www.ias.edu/people/faculty-and-emeriti/deligne`
5. Freitag, Eberhard and Kiehl, Reinhardt, *Etale Cohomology and the Weil Conjectures*, Ergebnisse der Mathematik under ihrer Grenzgebeite 3. Folge - Band 13, Springer-Verlag (1988).
6. Discussions with Alok Maharana and Yashonidhi Pandey.
7. The last section is heavily dependent on some understanding gained of Deligne's second proof of Weil conjectures during a week long seminar on this topic at TIFR conducted with V. Srinivas and a host of other participants.

2 Biography

Early Studies
Pierre Deligne was born on 3rd October 1944 in a district of Brussels, Belgium.
He did his schooling in Brussels and joined the Libre University of Brussels at the age of 18.
In his third year, he was deputed to the Ecole Normale Superieure of Paris, from which he received his Licence for (teaching) Mathematics the same year.

Doctoral Studies

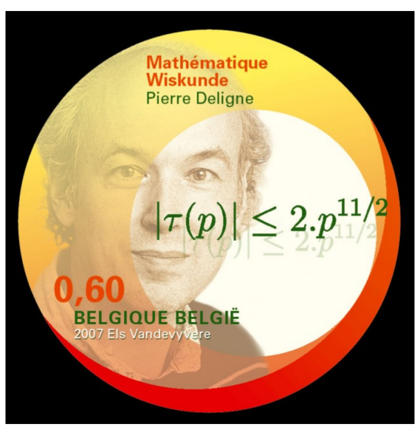


Continued at Libre University of Brusells for his doctoral studies and became a *Junior Scientist* when he was 23; however, he was actually spending most of his time at the Institut Haute Etudes Scientifique, France. He worked on his PhD under the guidance of **Alexander Grothendieck**. He was awarded his Doctorate by the Free University of Brussels at the age of 24.

Positions and Honours
Visiting member of the IHES for two years and then a permanent member when he was 26.
After spending 14 years at the IHES, Deligne moved to the Institute for Advanced Study when he was 40.
Some "major" honours:

- Plenary Speaker at the ICM (1974)
- Fields Medal (1978)
- Wolf Prize (2008)
- Abel Prize (2013)

Made a Viscount (like Padma Vibhushan) in 2006 by the King of Belgium.



Real Honours
He is described by two great Mathematicians and Abel Prize winners, Jean-Pierre Serre (2003, First Abel Prize) and John Tate (2010) as possibly the greatest mathematician ever.

Personal Snippets

- He doesnt own a car and bicycles everywhere in Princeton.
- He doesn't own a suit (at least he has never been seen wearing one!).
- When he wants to get mathematics done, he stretches out on a couch surrounded by his calculations and relevant papers.
- In winter, he builds an igloo in his backyard and spends a few nights sleeping outside.
- In summer, he maintains a kitchen garden, growing potatoes, tomatoes, raspberries, gooseberries, leeks, basil, parsley, tarragon, chives.
- During treks in Rajmachi in Western India, he has been known to sleep on the embankments of the fort at night.

Lecturer, Teacher, Colleague

- He prepares his talks meticulously, but speaks too softly and writes too small for the audience.A standard instruction from his audience has been:Write Larger and Speak Louder.
- He never “sells” his mathematics and adopts a style of letting his mathematics “speak for itself”.
- He prepares detailed notes on everything he studies and is always willing to share these notes with others.
- He is gentle, soft spoken and contemplative during discussions: “there is no complex facade to get through”.
- He’s a ”nice guy” according to all colleagues and staff at IAS, Princeton.

Mathematical Research
Broadly speaking Deligne’s area of research can be explained as:

Algebraic Geometry, Algebraic Topology and Number Theory

Primary Interest “the basic objects of arithmetical algebraic geometry—Motives”.The concept of ”Motives” or ”Motifs” was first mooted by Alexander Grothendieck.

Topics Studied algebraic varieties, L-functions, Shimura varieties, trigonometrical sums, linear differential equations and their monodromy, representations of finite groups, quantization deformation, supersymmetry, Hodge theory, modular forms, Galois representations.

Working with Others

Collaborations Deligne-Mumford, Deligne-Serre, Deligne(Grothendieck), Deligne-Lusztig, Deligne-Mostow, Deligne-Illusie,Deligne-Katz, Deligne-Langlands, Deligne-Milne-Ogus-Shih,Delign-Beilinson-Bernstein-Gabber, Deligne-Griffiths-Morgan-Sullivan.

Expositions Griffiths, Shimura, Langlands, Herman, Patterson, Fulton,Saavedra-Rivano, Quillen, Grothendieck, Milnor, Witten.

Publications

- His first published paper in the Bulletin of the Belgian Mathematical Society in 1966 (at age 22) on congruences satisfied by the number of subgroups of a group of order p^k.
- Later in the same year, he explained the "correct way" of talking about cohomology with proper support, in an appendix to the notes by Hartshorne on "Residues and Duality" of a course by Grothendieck.
- First major publication on the use of the Lefschetz Theorem to provide a criterion for the degeneration of the Leray Spectral Sequence. In the Publications of the IHES at the age of 24.
- Proof of Weil conjectures in Publications of the IHES;First proof in 1974 and Second proof in 1980.
- Last listed publication (number 86) on the Princeton Home Page, is on the Representations of the symmetric group S_t when t is not a natural number, as part of the proceedings of the International Colloquium at TIFR in 2004 on Algebraic Groups and Homogeneous Spaces.

He continues to write papers, notes and is very active in Mathematical Research.
Finally, let us do some mathematics! Just as Gauss proved a large number of theorems, but is (perhaps) most famous for his proof of the quadratic reciprocity law, this is the "stand out" result of Deligne's work. Among other proofs of the quadratic reciprocity, the one using (what are now called) Gauss sums is the one that lead to a lot of later research (such as Kummer's proof of Fermat's Last Theorem for a number of cases). Similarly, Deligne's second proof of the Weil conjectures has exposed some of the key interactions between analytic number theory, algebraic number theory and geometry. So that is the proof we will look at here.
In the next section, we will give a brief explanation of the Weil Conjectures. In the following section, we will present some key ideas of Deligne's (second) proof of these conjectures.

3 Weil Conjectures

Diophantine Geometry is the study of the solutions in integers of a system of polynomial equations. To simplify our explanation let us take the just one equation:

$$V : X^5 + Y^5 = Z^5$$

a famous equation for which Fermat claimed (and Kummer proved) that there are no solutions other than when $XYZ = 0$. When we substitute $(X, Y, Z) = (2, 3, 7)$ we get the difference of the two sides to be $16532 = 2^2 \cdot 4133$ where 4133 is a prime. we can thus write

$$2^5 + 3^5 \equiv 7^5 \pmod{4133}$$

Following this it is easy to find solutions *modulo* primes.
When p is a prime, addition and multiplication modulo p are taken to mean that we always take the remainder of division by p after carrying out the relevant operation. It then turns out that we have subtraction and division as well! Thus the collection of integers between 0 and $p-1$ form a *field* which we denote by $\mathbb{F}_p$ — the field with p elements. One can look for other fields with finitely many elements and it turns out that the size of *any* finite field is some power p^k of a prime p.
The collection of solutions in a finite field $\mathbb{F}_{p^k}$ of a system V of equations as above, is a *finite* set $V(\mathbb{F}_{p^k})$ — so we can count the number of elements $N_k = |V(\mathbb{F}_{p^k})|$ in it. So Andre Weil (following Emil Artin and Helmut Hasse), defined the following function that combines all these sizes

$$\zeta_p(V, s) = \exp\left(\sum_{k=1}^{\infty} \frac{N_k}{k}(p^{-s})^k\right)$$

Hasse had already shown that this function had some nice properties in the case of cubic curves (such as $E : X^3 + Y^3 = Z^3$). Weil was able to generalise these results. In the case of the equation V as given above

$$\zeta_p(V, s) = \frac{(1 - p^{-s})(1 - p^{2-s})}{P(p^{-s})}$$

where $P(T)$ is a *polynomial* of degree 6 with *integer* coefficients. Moreover, by finding its roots over complex numbers, one can write

$$P(T) = \prod_i (1 - a_i T)(1 - b_i T)$$

where a_i and b_i are conjugate complex numbers each having absolute value (length) $p^{1/2}$.
Moreover, Weil was able to generalise this to all systems V of equations where the solutions of V over the field of complex numbers forms a *surface* of genus g — which we think of as a *medhu vada* with not one but g holes. In that case the polynomial P has degree $2g$!
This led Weil to conjecture that:

> Counting the number of solutions of a system of equations V over a finite field gives us the "shape" (topology) of the same system when it is solved over the field of complex numbers!

The more precise conjecture is that for systems V of Diophantine equations, the associated function $\zeta_p(V, s)$ that counts its solutions in $\mathbb{F}_{p^k}$ has a decomposition as:

$$\zeta_p(V, s) = \prod_{m=0}^{\dim(V)} P_m(p^{-s})^{(-1)^m}$$

where the right-hand side refers to the geometry of the solutions $V(\mathbb{C})$ of V over the field of complex numbers in the following ways:

- $\dim(V)$ is the dimension of the space $V(\mathbb{C})$.
- $P_m(T)$ are polynomials whose *degree* is the m-th Betti number of the space $V(\mathbb{C})$. (Poincaré and Betti defined some numbers β_m that represent the "holey"-ness of spaces, where m refers to the "dimension of the hole"s; β_1 is $2g$ for a.)
- The polynomial $P_m(T)$ is a polynomial with integer coefficients which can be split using complex numbers

  $$P_m(T) = \prod_i (1 - a_{m,i} T)$$

  where these complex numbers $a_{m,i}$ are of absolute value $p^{m/2}$.

The last item does not refer to the geometry of $V(\mathbb{C})$ in an obvious way! Indeed, it is this item that is the most difficult aspect of the Weil Conjectures. In writing this last part, Weil was (almost certainly) subconsciously triggered by his reading of Ramanujan's conjecture on the τ function. (The first explicit connection was made by Langlands based on earlier idea of Rankin.)

Digression of sorts: Ramanujan had extensively studied the function $\tau(n)$ defined by the generating function

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n \geq 1} (1 - q^n)^{24}$$

In particular, he had made the conjecture that

$$|\tau(p)| \leq 2p^{11/2}$$

We note that if $P(T) = 1 - mT + nT^2$ (with m, n integers) can be written as

$$P(T) = (1 - aT)(1 - bT) \text{ where } a \text{ and } b \text{ are complex such that } |a| = |b| = p^{11/2}$$

then $n = \pm p^{11}$ and $|m| \leq p^{11/2}$. Since Hasse's expression for a cubic curve E had exactly the same structure with 11 replaced by 1, this was an intriguing connection!

Now, the first two parts of Weil's conjectures were essentially proved by Grothendieck and Dwork. However, without the last part, this was mostly useless since there is no way to separate out the factors $P_m(p^{-s})$ without being able to separate the roots and poles of $\zeta_p(V, s)$. Grothendieck proposed his (conjectural) theory of motives as a way of explaining why the third statement *has* to be true for geometric reasons. He then outlined a number of ideas which, when carried out individually, would complete the picture. In fact, Grothendieck's ideas are being explored to this day but the complete picture as needed for his proposed proof of the Weil Conjectures has not yet emerged!

There were may sceptics who said that this enormous framework, which itself depended on pages and pages of more and more abstract mathematics, seemed far removed from many accessible problems in number theory, algebra and geometry. It was Deligne's proof of the Weil conjectures, combining many of Grothendieck's ideas with some "tricks" (which Grothendieck abhorred), that convinced may of the sceptics. In fact, by the time I started graduate studies in the 1980's, it was evident that real progress in many problems in algebraic number theory and algebraic geometry could be made by those who were well-versed in Grothendieck's version of these subjects. The primary reason for this conviction was Deligne's proof of the Weil conjectures. So we need to understand how Deligne achieved this miracle, even though his *guru* Grothendieck didn't believe in miracles!

4 Idea of Proof

4.1 Weil's ideas

One of the ideas of Emmy Noether that was carried through by Alexandroff was to think of the Betti numbers as the dimensions [1] of linear spaces. There are actually linear spaces $H^m(V)$ associated with $V(\mathbb{C})$ of dimension β_m. It then seems natural to think of $P_m(T)$ as the *characteristic polynomial* $\det(1 - F_m T)$ of a linear transformation F_m from $H^m(V)$ to itself. Furthermore, it is natural to think of F_m as the result of a transformation F of V to itself.

Weil saw that there is indeed a candidate for this transformation — the Frobenius endomorphism F. The fixed points of F^k are precisely $V(\mathbb{F}_{p^k})$. Lefschetz proved a fixed point theorem which would allow one to calculate the fixed points of an endomorphism in terms of its action on $H^m(V)$. The Weil conjectures (except the last part about the roots!) are just a re-statement of Lefschetz fixed point theorem!

So why didn't Weil state a theorem rather than conjectures? There are some problems:

- The endomorphism F is not defined on V but on $V \pmod p$ which is the system V together with the equation $p = 0$!
- The vector spaces $H^m(V)$ are associated with $V(\mathbb{C})$ while the points we are looking for are in $V(\mathbb{F}_{p^k})$. There is no natural relation between the fields $\mathbb{C}$ and $\mathbb{F}_{p^k}$, since $p = 0$ in the latter.
- Even if all this were to work, why would the eigenvalues of F_m have the property that they have absolute value $p^{m/2}$?

Weil was able to take put these ideas together in the case when $d = 1$. In other words, Weil proved his conjectures for *curves*.

[1] I learned of this idea from Ramanan who told me, "Whenever you see a number think dimension of a linear space"; this replacement of a finite set (of states) by a linear combination of its elements (linear superposition of states) is (first) quantization!

4.2 Grothendieck's ideas

A fundamental idea in mathematical proof is that of *induction*. In the geometric setup, the natural induction is by dimension. Given a system V of equations, the complex solutions $V(\mathbb{C})$ form a manifold of dimension $2d$ for some d (the 2 comes from the fact that $\mathbb{C}$ are represented by the Gaussian *plane*); d is called the dimension of V.

Given any V, there is a system W of dimension 1 and a morphism (substitution rule) $f : V \to W$ which allows us to think of V as made up of the fibres, for each t in W:

$$V_t = f^{-1}(t) = \{v \in V | f(v) = t\}$$

Now, the fibres have dimension $d - 1$ and patching data that allows us to piece them together is W which of dimension 1. So there is scope for an inductive argument.

Patching the linear spaces $H^m(V_t)$ together as t varies over W to get to $H^{m+\{0,1,2\}}(V)$ comes from an idea of Leray that Grothendieck developed further into what is now called the Grothendieck-Leray spectral sequence.

An important difficulty to overcome is to make an analogous construction for $V \pmod p$. A decade long development of a key idea of Grothendieck called *étale cohomology*, was carried out in collaboration with many of his students and other mathematicians in a seminar called SGA (Seminaire Geometrie Algebrique). Grothendieck introduced the notion of (chain complexes of) étale sheaves which could be "pushed forward" and "pulled back" under morphisms. Combining morphisms required the use of spectral sequences (like the Grothendieck-Leray spectral sequence). This was called the "formalism of six functors".

One of the beautiful aspects of étale cohomology was that it combined two theories that had only *seemed* similar before. On the one hand one had the theory of Galois which studied how groups operate on solutions of algebraic equations. On the other hand one had a geometric theory of *covering spaces*. Grothendieck was able to convert this analogy into a combined theory which made Galois transformations like the Frobenius endomorphism operate on geometric objects like $H^m(V)$.

One difficulty was that this did not produce $H^m(V \pmod p)$ but only something like $H^m(V \pmod p) \pmod N$ for N and p co-prime. In spite of this, the fact that the Frobenius endomorphism does act on these étale cohomology groups made it possible to prove the first two Weil conjectures. This was done by Grothendieck. Dwork also proved the rationality of $\zeta_p(V, s)$ following a different approach.

4.3 Deligne's ideas

The somewhat mysterious third part of Weil conjectures remained. To complete this, Deligne had to combine three different ideas.

First, he introduced the notion of "mixed Weil sheaves". An étale sheaf is made by patching together objects of the form $L \pmod N$ where L is a lattice. A linear map from a lattice L to itself can be represented by a matrix M whose coefficients are integers. A mixed Weil sheaf is an étale sheaf such that the Frobenius endomorphism of $L \pmod N$ can be represented by a matrix M whose characteristic polynomial can be split as $\prod_i (1 - a_i T)$ where he absolute values of a_i are of the form $p^{m_i/2}$ for integers m_i.

Deligne realised that to prove the Weil conjectures, he would need to show that mixed Weil sheaves also have the "push forward" and "pull back" formalism of six functors. In addition, one would need to prove the the final result was not "mixed" but "pure", in the sense that all the m_i have the same value.

So why not use "pure Weil sheaves" from the beginning? The reason is that it the behaviour of the Frobenius is "pure" only when the V is smooth. In the inductive step mentioned above, one must find a way to deal with the case when V_t is not smooth — which it will always be for a finite number of parameters t in W. Deligne then brought in a critical idea (a pun for the *cognoscenti*!) due to Lefschetz of choosing a morphism $f : V \to W$ where the non-smooth (singular) fibres V_t have very mild singularities called ordinary double points. Lefschetz had studied such morphisms (called Lefschetz pencils in his honour) in the context of $V(\mathbb{C}) \to W(\mathbb{C})$; i. e. over complex numbers. Deligne wished to show that this resulted in a "mixed Weil" structure for $H^m(V_t)$ when V_t had an ordinary double point singularity. He could prove this provided he had proved (by induction) that the $H^m(V)$ are *pure*.

To complete the argument, he needed to prove that "irreducible" mixed Weil sheaves on a curve have "pure" étale cohomologies. Deligne he recognised that this could be deduced from the non-vanishing of a generating function associated with such a sheaf. This generating function, which is similar to the $\zeta_p(V, s)$ introduced earlier is called an L-function. The non-vanishing of L-functions and ζ functions has played an important role in *analytic* number theory and formed the key step in the proof by Hadamard and de la Vallee-Poussin of the proof of the prime number theorem (conjectured by Gauss). The idea of using non-vanishing to prove such theorems goes back to Riemann[2] when he sketched how the Riemann hypothesis could be used to prove the prime number theorem. Deligne was able to re-interpret the method of Hadamard and de la Vallee-Poussin in a novel way to also prove the non-vanishing of his L-function.

To summarise, the contribution of Deligne was to recognise the need of "mixed Weil sheaves" and the use of Lefschetz pencils (which may have occurred to others). Both of these are not "tricks" in the sense of Grothendieck and can be fit neatly into a theory. However, the amazing and tricky aspect was his interpretation and use of the method of Hadamard and de la Vallee-Poussin to prove that the logarithms of the eigenvalues were sufficiently far way from the other powers of $p^{1/2}$ in order to conclude the proof of purity.

[2] This idea of Riemann was explained to me by Balasubramanian in a seminar he gave on his topic in 1984 at the TIFR.