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Remarks on absolute de Rham and absolute Hodge cycles

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Let $X$ be a smooth proper variety over a field $k$ of characteristic zero. For any embedding $\sigma$ of $k$ into the field of complex numbers $\mathbb{C}$, the $\mathbb{C}$ valued points of $X \otimes_\sigma \mathbb{C}$ form a complex manifold denoted by $X_\sigma$. By base change for the de Rham cohomology $H^j_{DR}(X/k) \otimes_\sigma \mathbb{C} = H^j_{DR}(X \otimes_\sigma \mathbb{C}/\mathbb{C})$ and by the GAGA principle one has an isomorphism $I_\sigma$ from $H^j_{DR}(X/k) \otimes_\sigma \mathbb{C}$ to the Betti cohomology $H^j_B(X_\sigma, \mathbb{C})$ ([5], p. 96).

An element of the $\mathbb{Q}$ Chow group $CH^i(X) \otimes_\mathbb{Z} \mathbb{Q}$ has a de Rham class
$$\alpha \in F^i H^{2i}_{DR}(X/k) = H^{2i}(X, \Omega^{\geq i}_X/k) \subset H^{2i}_{DR}(X/k)$$
such that for all embeddings $\sigma : k \to \mathbb{C}$
$$I_\sigma(\alpha) \in I_\sigma(F^i H^{2i}_{DR}(X/k) \otimes_\sigma \mathbb{C}) \cap H^{2i}_B(X_\sigma, \mathbb{Q}).$$
So $\alpha$ is an absolute Hodge cycle, a notion defined by Deligne [3], §2, which we slightly modify, as we are only interested here in de Rham cohomology (see [3], open question 2.2).

**Definition 1** A class $\alpha \in F^i H^{2i}_{DR}(X/k)$ is said to be an absolute Hodge cycle if for all embeddings $\sigma : k \to \mathbb{C}$, $I_\sigma(\alpha)$ lies in $H^{2i}_B(X_\sigma, \mathbb{Q})$.

On the other hand, such an algebraic cycle has an absolute de Rham class in $H^{2i}(X, \Omega^{\geq i}_X/Q)$. In fact, there is an absolute differential
$$d\log : \mathcal{O}_X^\times \to \Omega^{\geq 1}_{X/Q}$$ [1]
inducing an absolute differential
\[ d \log : K_i^M \rightarrow \Omega^{\geq i}_{X/Q} \]
where \( K_i^M \) is the Zariski sheaf of Milnor \( K \) theory. As \( CH^i(X) \otimes \mathbb{Z} \mathbb{Q} = H^i(X, K_i^M) \) ([9], théorème 5), \( d \log \) induces the absolute de Rham cycle class map
\[ CH^i(X) \otimes \mathbb{Z} \mathbb{Q} \xrightarrow{\psi} \mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/Q}). \]

One composes this map with
\[ \mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/Q}) \rightarrow \mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/k}) = F^iH^{2i}_{DR}(X/k) \]
to obtain the de Rham cycle class map. As we don’t have a reference for this, we indicate how to prove it by base change
\[ F^iH^{2i}_{DR}(X/k) \otimes \sigma \mathbb{C} = F^iH^{2i}_{DR}(X \otimes \sigma \mathbb{C}/\mathbb{C}), \]
so it is enough to handle \( k = \mathbb{C} \), in which case the compatibility is proven in [2], (2.2.5.1) and (2.2.5.2) for \( i = 1 \). For \( i > 1 \), resolving the structure sheaf of an effective cycle by vector bundles, and for a given vector bundle, computing its Chern classes on the Grassmannian bundle
\[ G \rightarrow X, \quad \pi^* : F^iH^{2i}_{DR}(X/\mathbb{C}) \rightarrow F^iH^{2i}_{DR}(G/\mathbb{C}), \]
one reduces the compatibility to the case \( i = 1 \).

Remark 2 The existence of the absolute de Rham cycle class is proven in great generality in [10] when \( X \) is singular. In fact, this class is convenient to formulate some questions. For example, its injectivity for a surface \( X \) over \( k = \mathbb{C} \) would imply Bloch’s conjecture when \( H^2(X, \mathcal{O}_X) = 0 \).

At any rate, the existence of \( \psi \) motivates the following

Definition 3 A class \( \alpha \in F^iH^{2i}_{DR}(X/k) \) is said to be an absolute de Rham cycle if it lies in the image of \( H^{2i}_{DR}(X/\mathbb{Q}) \) in \( \mathbb{H}^{2i}_{DR}(X/k) \).

We denote by \( \nabla : H^j_{DR}(X/k) \rightarrow \Omega^1_{k/\mathbb{Q}} \otimes_k H^j_{DR}(X/k) \) the Gauss-Manin connection for the smooth morphism \( X \rightarrow \text{Spec } k \) of schemes over \( \text{Spec } \mathbb{Q} \).

Proposition 4 The sequence
\[ H^1_{DR}(X/k) \rightarrow H^1_{DR}(X/k) \xrightarrow{\nabla} \Omega^1_{k/\mathbb{Q}} \otimes H^1_{DR}(X/k) \]
is exact.

Proof. The sequence is obviously a complex. Let \( k_0 \subset k \) be the field of definition of \( X \). One has \( X = X_0 \otimes_{k_0} k \), where \( X_0 \) is smooth proper over \( k_0 \), and \( k_0 = \mathbb{Q}(S_0) \) for a smooth affine variety \( S_0 \) over \( \mathbb{Q} \), such that there is
a smooth proper map $f_0 : \mathcal{X}_0 \to S_0$ with $\mathcal{X}_0 \otimes_{\mathcal{O}_{S_0}} k_0 = X_0$.

As $H_{DR}(X_0/k_0)$ is a finite dimensional $k_0$ vector space, any

$$\alpha \in H^{1}_{DR}(X/k) = H^{1}_{DR}(X_0/k_0) \otimes_{k_0} k$$

lies in $H^{1}_{DR}(X_0/k_0) \otimes_{k_0} \mathbb{Q}(S)$, where $k_0 \subset \mathbb{Q}(S) \subset k$ and $S$ is a smooth affine variety mapping to $S_0$. If $x \in \text{Ker}\nabla$, then $x$ lies in the kernel of

$$H_{DR}^{1}(X_0 \otimes_{k_0} \mathbb{Q}(S)/\mathbb{Q}(S)) \to \Omega^{1}_{\mathbb{Q}(S)/\mathbb{Q}} \otimes H_{DR}^{1}(X_0 \otimes_{k_0} \mathbb{Q}(S)/\mathbb{Q}(S))$$

and to prove exactness, one has to see that

$$\alpha \in \text{Im} \ (H_{DR}^{1}(X_0 \otimes_{k_0} \mathbb{Q}(S)/\mathbb{Q}(S)) \to H_{DR}^{1}(X_0 \otimes_{k_0} \mathbb{Q}(S)/\mathbb{Q}(S))).$$

Denote by $f : \mathcal{X} = \mathcal{X}_0 \times_{S_0} S \to S$ the smooth proper morphism obtained by base change $S \to S_0$ of $f_0$. Making $S$ smaller, one may assume that there is

$$\beta \in \text{Ker}(H_{DR}(\mathcal{X}/S) \xrightarrow{\nabla} \Omega^{1}_{\mathcal{X}/\mathbb{Q}} \otimes H_{DR}(\mathcal{X}/S))$$

such that $\beta \otimes_{\mathcal{O}_{\mathcal{X}}} \mathbb{Q}(S) = \alpha$, and one wants to show that $\beta \in \text{Im} \ H^{1}_{DR}(\mathcal{X}/\mathbb{Q})$.

On $\Omega^a_{\mathcal{X}/\mathbb{Q}}$ one considers the filtration by the subcomplexes $f^{*}\Omega^{\geq a}_{\mathcal{S}/\mathbb{Q}} \land \Omega^{\leq a}_{\mathcal{X}/\mathbb{Q}}$. It defines a spectral sequence

$$E_{1}^{ab} = \Omega^{a}_{\mathcal{S}/\mathbb{Q}} \otimes H_{DR}^{b}(\mathcal{X}/S)$$

converging to $H^{a+b}_{DR}(\mathcal{X}/\mathbb{Q})$, whose $d_1$ differential is the Gauss-Manin connection $\nabla$. As $S$ is affine, one has

$$E_{2}^{ab} = \mathbb{H}^{a}(S, \Omega^{*}_{\mathcal{S}/\mathbb{Q}} \otimes H_{DR}^{b}(\mathcal{X}/S)).$$

We now consider the analytic varieties $S_{an} = (S \otimes_{\mathbb{Q}} \mathbb{C})_{an}$, $\mathcal{X}_{an} = (\mathcal{X} \otimes_{\mathbb{Q}} \mathbb{C})_{an}$. The corresponding spectral sequence

$$E_{2,an}^{ab} = \mathbb{H}^{a}(S_{an}, \Omega^{*}_{\mathcal{S}_{an}} \otimes H_{DR}^{b}(\mathcal{X}_{an}/S_{an}))$$

which abuts to $\mathbb{H}^{a+b}(\mathcal{X}_{an}, \Omega^{*}_{\mathcal{X}_{an}}) = H^{a+b}(\mathcal{X}_{an}, \mathbb{C})$. This spectral sequence is, according to Deligne ([11], (2.77) and (15.6)) the Leray spectral sequence, and by [2], (4.1.1) (i), it degenerates at $E_2$.

On the other hand, by the regularity of the Gauss-Manin connection, one has

$$E_{2,an}^{ab} = \mathbb{H}^{a}(S \otimes_{\mathbb{Q}} \mathbb{C}, \Omega^{*}_{\mathcal{S} \otimes_{\mathbb{Q}} \mathbb{C}/\mathbb{C}} \otimes H_{DR}^{b}(\mathcal{X} \otimes_{\mathbb{Q}} \mathbb{C}/S \otimes_{\mathbb{Q}} \mathbb{C}))$$

([1], (6.2) and (7.9)).

This implies that $(E_{1}^{ab}, d_1) \otimes_{\mathbb{Q}} \mathbb{C}$ degenerates at $E_2$, and so does $(E_{1}^{ab}, d_1)$. In particular

$$H^{1}_{DR}(\mathcal{X}/\mathbb{Q}) = H^{0}(S, \Omega^{*}_{\mathcal{S}/\mathbb{Q}} \otimes H^{1}_{DR}(\mathcal{X}/S))$$

$$= \text{Ker}(H^{0}(S, H_{DR}(\mathcal{X}/S)) \to H^{0}(S, \Omega^{1}_{\mathcal{S}/\mathbb{Q}} \otimes H^{1}_{DR}(\mathcal{X}/S))).$$
This proves the required exactness by base change to \( \mathbb{Q}(S) \).

**Remark 5** In fact, even if \( S \) is not affine, there is a Leray spectral sequence for the de Rham cohomology \([7](3.3)\), which again degenerates at \( E_2 \) by the comparison between the Leray spectral sequences for the Betti and the de Rham cohomologies, and the regularity of Gauss-Manin. For more on this, see [8].

**Corollary 6** If \( \alpha \) is an absolute Hodge cycle, then it is an absolute de Rham cycle.

**Proof.** By [3] (2.5), we know that \( \nabla \alpha = 0 \), where \( \nabla \) is as in (4) for \( j = 2i \). Then we apply (4).

**Corollary 7** If \( \alpha \) is an absolute de Rham cycle such that \( I_\sigma(\alpha) \in H^{2i}_B(X_\sigma, \mathbb{Q}) \) for some embedding \( \sigma : k \to \mathbb{C} \), then \( \alpha \) is an absolute Hodge cycle.

**Proof.** In fact, this is [3] (2.6). More precisely, choose \( S \) as in the proof of 4 and \( \beta \in H^{2i}_D(X/S) \) restricting to \( \alpha \). The embeddings \( \mathbb{Q}(S) \to k \to \mathbb{C} \) define a \( \mathbb{C} \)-valued point of \( S \), which we still denote by \( \sigma \), such that \( \beta(\sigma) \in H^{2i}(X_{an}, \mathbb{Q}) \). The image \( \beta(\sigma) \) of \( \beta \) in

\[
H^0(S_{an}, R^{2i}f_\ast \mathbb{C}) = H^{2i}(X_{an}, \mathbb{C})^\pi_1(S_{an, \sigma})
\]

lies in

\[
H^0(S_{an}, R^{2i}f_\ast \mathbb{Q}) = H^{2i}(X_{an}, \mathbb{Q})^\pi_1(S_{an, \sigma}).
\]

Therefore \( \beta|_{X_{an, \sigma}} \) is rational for all \( s \), in particular for those \( s \) coming from an embedding \( \sigma : k \to \mathbb{C} \).

**Remark 8** An advantage, if any, to adopt the language of absolute de Rham cycles consists of dividing the question of whether \( \alpha \) is absolute Hodge or not into two steps:

First of all \( \alpha \) must be in

\[
H^{2i}_D(X_0/k_0) \otimes_{k_0} k_0^{alg} = \ker H^{2i}_D(X_0/k_0) \otimes_{k_0} k \to \Omega^1_{k_0/k_0} \otimes_{k_0} H^{2i}_D(X_0/k_0),
\]

where \( k_0^{alg} \) is the algebraic closure of \( k_0 \) in \( k \).

Secondly \( \alpha \) must be in

\[
\ker H^{2i}_D(X_0/k_0) \otimes_{k_0} k_0^{alg} \to \Omega^1_{k_0/k_0} \otimes_{k_0} H^{2i}_D(X_0/k_0) \otimes_{k_0} k_0^{alg}.
\]

On the other hand, we have seen that if \( \alpha \in F^iH^{2i}_D(X/k) \) is the class of an algebraic cycle, then not only it is an absolute de Rham cycle, but also it is coming from \( \mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/Q}) \).

Let \( f : X \to S, \beta \in F^iH^{2i}_D(X/S) = H^0(S, R^jf_\ast \Omega^{\geq i}_{X/S}) \), such that \( \beta \otimes_{\mathbb{Q}(S)} k = \alpha \in \cdots \)
$F^iH^j_{DR}(X/k)$ as in the proof of 4. Let $f_c : X_c \to S_c$ be the smooth proper morphism obtained from $f$ by base change $O_S \otimes_Q C$, and $\beta_c$ be $\beta \otimes_Q C$. Let $\overline{f}_C : \overline{X}_C \to \overline{S}_C$ be a compactification of $f_C$ such that $\Sigma = \overline{S}_C - S_C$, $D = \overline{f}_C^{-1}(\Sigma)$ are normal crossing divisors and $\overline{X}_C$ is smooth.

**Definition 9** A class $\alpha \in F^iH^j_{DR}(X/k)$ is said to be of moderate growth if for some $(\beta, \overline{f}_C)$ as above, it verifies

$$(*) \quad \beta_c \in H^0(\overline{S}_C, R^j\overline{f}_C^*\Omega^\ge_{\overline{X}_C/\overline{S}_C}(\log D)) \subset H^0(S_C, R^j f_C^*\Omega^\ge_{X_c/S_c})$$

**Remark 10** The definition 9 does not depend on the couple $(\beta, \overline{f}_C)$ choosen. In fact, take $(\gamma, g)$ with $g : Y \to T$, $\mathbb{Q}(T) \subset k$, $Y \otimes_{\mathbb{Q}(T)} k = X$, $\gamma \otimes_{\mathbb{Q}(T)} k = \alpha$. Then considering in $k$ a function field $\mathbb{Q}(U)$ containing $\mathbb{Q}(S)$ and $\mathbb{Q}(T)$, one has base changes $\sigma : U \to S$, $\tau : U \to T$, $f_U : X_U = X \times_S U \to U$, $g_U : Y_U = Y \times_T U \to U$, such that there is an isomorphism $\iota : X_U \to Y_U$, with $g_U \circ \iota = f_U$, $\iota^*(\gamma \otimes_{\mathbb{Q}(T)} \mathcal{O}_U) = \beta \otimes_{\mathbb{Q}(S)} \mathcal{O}_U$, for $U$ small enough. As $\beta_c$ fulfills $(*)$ on $\overline{S}_C$, it fulfills $(*)$ on any blow up $\overline{\sigma}_C : \overline{U}_C \to \overline{S}_C$ such that a commutative diagram exists

$$\begin{array}{ccc}
X_{U,C} & \longrightarrow & \overline{X}_C \\
\overline{U}_C \downarrow \quad \overline{\sigma}_C \quad \downarrow \overline{\tau}_C \\
\overline{U}_C & \longrightarrow & \overline{S}_C
\end{array}$$

with the properties: $\overline{\sigma}^{-1}\Sigma$, $\Delta = \overline{f}_{U,C}^{-1}\overline{\sigma}^{-1}\Sigma$ are normal crossing divisors, $X_{U,C}$ and $\overline{U}_C$ are smooth. Choose $\overline{U}_C$ such that $\tau$ extends to $\overline{\sigma_c} : \overline{U}_C \to \overline{T}_C$, with a commutative diagram

$$\begin{array}{ccc}
X_{U,C} & \longrightarrow & \overline{Y}_C \\
\overline{U}_C \downarrow \quad \overline{\sigma}_C \quad \downarrow \overline{\tau}_C \\
\overline{U}_C & \longrightarrow & \overline{T}_C
\end{array}$$

with the same properties as above. One has now

$$H^0(\overline{U}_C, R^j\overline{f}_{U,C}^*\Omega^\ge_{X_{U,C}/Y_{U,C}}(\log \Delta)) = H^0(\overline{T}_C, R^j\overline{g}_{\overline{U,C}}^*\Omega^\ge_{X_{\overline{U},/T_{\overline{U}}} (log \overline{\sigma}_C^{-1}(\overline{T}_C - \overline{T}_C)})$$

[6], 4.13.

This implies in particular that classes of moderate growth build a $k$ subvectorspace of $F^iH^j_{DR}(X/k)$.

**Notation 11** We denote this subvectorspace by $F^iH^j_{DR}(X,k)^{log}$, and by $\mathbb{H}^j(X, \Omega^\ge_{X/k})^{log}$ its inverse image in $\mathbb{H}^j(X, \Omega^\ge_{X/k})$.

**Theorem 12** The sequence

$$\mathbb{H}^j(X, \Omega^\ge_{X/k})^{log} \longrightarrow F^iH^j_{DR}(X/k)^{log} \longrightarrow \Omega^1_{k/Q} \otimes F^{i-1}H^j_{DR}(X/k)$$

6
is exact.

Proof. We have to prove that if \( \alpha \in \text{Ker} \nabla \), then it lies in the image of \( \mathbb{H}^j(X, \Omega^i_{X/Q}) \).

With the notations as above,
\[
\beta_C \in H^0(S_C, \Omega^i_{S_C}(\log \Sigma) \otimes R^j f_{C *} \Omega^j_{X_{C/S_C}}(\log D)).
\]

This group is the \( E_2^{0j} \) term of a spectral sequence converging to \( \mathbb{H}^j(X, \Omega^i_{X/Q}) \) and defined as in [7] (3.3) on the complex \( \Omega^j_{X_{C/S_C}}(\log D) \).

One has
\[
E_2^{ab} = H^a(S_C, \Omega^i_{S_C}(\log \Sigma) \otimes R^b f_{C *} \Omega^j_{X_{C/S_C}}(\log D)).
\]

By [6] (0.4) and its analogue in characteristic zero [4] (2.7), \( E_2^{ab} \) injects into
\[
H^a(S_C, \Omega^i_{S_C}(\log \Sigma) \otimes R^b f_{C *} \Omega^j_{X_{C/S_C}}(\log D)),
\]

which is just \( H^a(S_{an}, R^b f_{C *}(\mathbb{C})) \) by [1] II, §6.

Thus the spectral sequence degenerates at \( E_2 \), and \( \beta_C \) comes from \( \mathbb{H}^j(X, \Omega^i_{X/Q}) \).

In particular \( \beta_C \) comes from \( \mathbb{H}^j(X, \Omega^i_{X/Q}) \otimes \mathbb{C} \) and the image of \( \alpha \) in
\[
\frac{F^i H^j_{DR}(X/k)}{\text{Im } \mathbb{H}^j(X, \Omega^i_{X/Q}) \otimes \mathbb{C}} = \left( \frac{F^i H^j_{DR}(X/k)}{\text{Im } \mathbb{H}^j(X, \Omega^i_{X/Q})} \right) \otimes \mathbb{C}
\]

vanishes. Therefore \( \alpha \) lies in the image of \( \mathbb{H}^j(X, \Omega^i_{X/Q}) \).

Remark 13 If the transcendence degree of \( k \) is \( \leq 1 \), then of course the sequence
\[
\mathbb{H}^j(X, \Omega^i_{X/Q}) \longrightarrow F^i H^j_{DR}(X/k) \rightarrow \Omega^1_{k/Q} \otimes F^{i-1} H^j_{DR}(X/k)
\]
is trivially exact. But if the transcendence degree of \( k \) is higher, it is not clear why an absolute Hodge cycle has to be a moderate absolute de Rham cycle.

More generally, one can consider a \( k \) subvectorspace \( V \) of \( H^j_{DR}(X/k) \), such that \( I_\sigma(V \otimes_\mathbb{C} \mathbb{C}) \) is a Hodge substructure of \( H^j_{DR}(X_\sigma, \mathbb{C}) \). In the light of the above results, one can examine the following questions.

Question 14 Is \( V \) stable under the Gauss-Manin connection?

For this, one would like \( I_\sigma^{-1}[I_\sigma(V \otimes_\mathbb{C} \mathbb{C}) \cap H^j_{DR}(X_\sigma, \mathbb{Q})] \) to lie in \( V \) and to be independent of \( \sigma \).

If so, then \( V \) defines a vector bundle \( W \) with a flat connection on \( S \), where \( S \) is defined as in 4 such that \( V = W \otimes_{\mathbb{Q}(S)} k \), \( W \subset H^j_{DR}(X_0/k_0) \otimes_{k_0} \mathbb{Q}(S) \). Then \( \mathcal{W}_{an} \) on \( S_{an} \) is generated by a local system \( \mathcal{F} \).
**Question 15** In the above situation, is the monodromy representation associated to $\mathcal{F}$ defined over $\mathbb{Q}$?

Again, one can split up 14 into two parts as in 8. Moreover, the knowledge of 14 does not imply the knowledge of 15.

**References**


[10] Srinivas, V.: Gysin maps and cycle classes for Hodge cohomology, preprint