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Remarks on absolute de Rham and absolute Hodge cycles

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Let X be a smooth proper variety over a field k of characteristic zero. For any embedding σ of k into the field of complex numbers $\mathbb{C}$, the $\mathbb{C}$ valued points of $X \otimes_\sigma \mathbb{C}$ form a complex manifold denoted by X_σ. By base change for the de Rham cohomology $H^j_{DR}(X/k) \otimes_\sigma \mathbb{C} = H^j_{DR}(X \otimes_\sigma \mathbb{C}/\mathbb{C})$ and by the GAGA principle one has an isomorphism I_σ from $H^j_{DR}(X/k) \otimes_\sigma \mathbb{C}$ to the Betti cohomology $H^j_B(X_\sigma, \mathbb{C})$ ([5], p. 96).

An element of the $\mathbb{Q}$ Chow group $CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ has a de Rham class

$$\alpha \in F^i H^{2i}_{DR}(X/k) = \mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/k}) \subset H^{2i}_{DR}(X/k)$$

such that for all embeddings $\sigma : k \to \mathbb{C}$

$$I_\sigma(\alpha) \in I_\sigma(F^i H^{2i}_{DR}(X/k) \otimes_\sigma \mathbb{C}) \cap H^{2i}_B(X_\sigma, \mathbb{Q}).$$

So α is an absolute Hodge cycle, a notion defined by Deligne [3], §2, which we slightly modify, as we are only interested here in de Rham cohomology (see [3], open question 2.2).

Definition 1 A class $\alpha \in F^i H^{2i}_{DR}(X/k)$ is said to be an *absolute Hodge cycle* if for all embeddings $\sigma : k \to \mathbb{C}$, $I_\sigma(\alpha)$ lies in $H^{2i}_B(X_\sigma, \mathbb{Q})$.

On the other hand, such an algebraic cycle has an absolute de Rham class in $\mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/\mathbb{Q}})$. In fact, there is an absolute differential

$$d\log : \mathcal{O}^*_X \longrightarrow \Omega^{\geq 1}_{X/\mathbb{Q}} \ [1]$$

inducing an absolute differential

$$d\log : \mathcal{K}_i^M \longrightarrow \Omega^{\geq i}_{X/\mathbb{Q}}\ [i]$$

where $\mathcal{K}_i^M$ is the Zariski sheaf of Milnor K theory. As $CH^i(X)\otimes_{\mathbb{Z}}\mathbb{Q} = H^i(X,\mathcal{K}_i^M)$ ([9], théorème 5), $d\log$ induces the absolute de Rham cycle class map

$$CH^i(X)\otimes_{\mathbb{Z}}\mathbb{Q} \xrightarrow{\psi} \mathbb{H}^{2i}(X,\Omega^{\geq i}_{X/\mathbb{Q}}).$$

One composes this map with

$$\mathbb{H}^{2i}(X,\Omega^{\geq i}_{X/\mathbb{Q}}) \longrightarrow \mathbb{H}^{2i}(X,\Omega^{\geq i}_{X/k}) = F^iH^{2i}_{DR}(X/k)$$

to obtain the de Rham cycle class map. As we don't have a reference for this, we indicate how to prove it. By base change $F^iH^{2i}_{DR}(X/k)\otimes_\sigma\mathbb{C} = F^iH^{2i}_{DR}(X\otimes_\sigma\mathbb{C}/\mathbb{C})$, so it is enough to handle $k=\mathbb{C}$, in which case the compatibility is proven in [2], (2.2.5.1) and (2.2.5.2) for $i=1$. For $i>1$, resolving the structure sheaf of an effective cycle by vector bundles, and for a given vector bundle, computing its Chern classes on the Grassmannian bundle $G\xrightarrow{\pi}X$, with $\pi^*: F^iH^{2i}_{DR}(X/\mathbb{C})\hookrightarrow F^iH^{2i}_{DR}(G/\mathbb{C})$, one reduces the compatibility to the case $i=1$.

Remark 2 The existence of the absolute de Rham cycle class is proven in great generality in [10] when X is singular. In fact, this class is convenient to formulate some questions. For example, its injectivity for a surface X over $k=\mathbb{C}$ would imply Bloch's conjecture when $H^2(X,\mathcal{O}_X)=0$.

At any rate, the existence of ψ motivates the following

Definition 3 A class $\alpha\in F^iH^{2i}_{DR}(X/k)$ is said to be an *absolute de Rham cycle* if it lies in the image of $H^{2i}_{DR}(X/\mathbb{Q})$ in $\mathbb{H}^{2i}_{DR}(X/k)$.

We denote by $\nabla : H^j_{DR}(X/k)\to\Omega^1_{k/\mathbb{Q}}\otimes_k H^j_{DR}(X/k)$ the Gauss-Manin connection for the smooth morphism $X\to\operatorname{Spec} k$ of schemes over $\operatorname{Spec}\mathbb{Q}$.

Proposition 4 *The sequence*

$$H^j_{DR}(X/\mathbb{Q}) \longrightarrow H^j_{DR}(X/k) \xrightarrow{\nabla} \Omega^1_{k/\mathbb{Q}}\otimes H^j_{DR}(X/k)$$

is exact.

Proof. The sequence is obviously a complex.
Let $k_0\subset k$ be the field of definition of X. One has $X = X_0\otimes_{k_0}k$, where X_0 is smooth proper over k_0, and $k_0=\mathbb{Q}(S_0)$ for a smooth affine variety S_0 over $\mathbb{Q}$, such that there is

a smooth proper map $f_0 : \mathcal{X}_0 \to S_0$ with $\mathcal{X}_0 \otimes_{\mathcal{O}_{S_0}} k_0 = X_0$.
As $H^j_{DR}(X_0/k_0)$ is a finite dimensional k_0 vector space, any

$$\alpha \in H^j_{DR}(X/k) = H^j_{DR}(X_0/k_0) \otimes_{k_0} k$$

lies in $H^j_{DR}(X_0/k_0) \otimes_{k_0} \mathbb{Q}(S)$, where $k_0 \subset \mathbb{Q}(S) \subset k$ and S is a smooth affine variety mapping to S_0. If $x \in \mathrm{Ker}\nabla$, then x lies in the kernel of

$$H^j_{DR}(X_0 \otimes_{k_0} \mathbb{Q}(S)/\mathbb{Q}(S)) \longrightarrow \Omega^1_{\mathbb{Q}(S)/\mathbb{Q}} \otimes H^j_{DR}(X_0 \otimes_{k_0} \mathbb{Q}(S)/\mathbb{Q}(S))$$

and to prove exactness, one has to see that

$$\alpha \in \ \mathrm{Im} \ \left(H^j_{DR}(X_0 \otimes_{k_0} \mathbb{Q}(S)/\mathbb{Q}) \longrightarrow H^j_{DR}(X_0 \otimes_{k_0} \mathbb{Q}(S)/\mathbb{Q}(S))\right).$$

Denote by $f : \mathcal{X} = \mathcal{X}_0 \times_{S_0} S \to S$ the smooth proper morphism obtained by base change $S \to S_0$ of f_0. Making S smaller, one may assume that there is

$$\beta \in \mathrm{Ker}(H^j_{DR}(\mathcal{X}/S) \xrightarrow{\nabla} \Omega^1_{S/\mathbb{Q}} \otimes H^j_{DR}(\mathcal{X}/S))$$

such that $\beta \otimes_{\mathcal{O}_S} \mathbb{Q}(S) = \alpha$, and one wants to show that $\beta \in \ \mathrm{Im} \ H^j_{DR}(\mathcal{X}/\mathbb{Q})$.

On $\Omega^\bullet_{\mathcal{X}/\mathbb{Q}}$ one considers the filtration by the subcomplexes $f^*\Omega^{\geq a}_{S/\mathbb{Q}} \wedge \Omega^{\bullet - a}_{\mathcal{X}/\mathbb{Q}}$. It defines a spectral sequence

$$E_1^{ab} = \Omega^a_{S/\mathbb{Q}} \otimes H^b_{DR}(\mathcal{X}/S)$$

converging to $H^{a+b}_{DR}(\mathcal{X}/\mathbb{Q})$, whose d_1 differential is the Gauss-Manin connection ∇. As S is affine, one has

$$E_2^{ab} = \mathbb{H}^a(S, \Omega^\bullet_{S/\mathbb{Q}} \otimes H^b_{DR}(\mathcal{X}/S)).$$

We now consider the analytic varieties $S_{an} = (S \otimes_{\mathbb{Q}} \mathbb{C})_{an}$, $\mathcal{X}_{an} = (\mathcal{X} \otimes_{\mathbb{Q}} \mathbb{C})_{an}$. The corresponding spectral sequence

$$\begin{aligned} E^{ab}_{2,an} &= \mathbb{H}^a(S_{an}, \Omega^\bullet_{S_{an}} \otimes H^b_{DR}(\mathcal{X}_{an}/S_{an})) \\ &= \mathbb{H}^a(S_{an}, \Omega^\bullet_{S_{an}} \otimes R^b f_* \Omega^\bullet_{\mathcal{X}_{an}/S_{an}}) \\ &= H^a(S_{an}, R^b f_* \mathbb{C}) \end{aligned}$$

which abuts to $\mathbb{H}^{a+b}(\mathcal{X}_{an}, \Omega^\bullet_{\mathcal{X}_{an}}) = H^{a+b}(\mathcal{X}_{an}, \mathbb{C})$. This spectral sequence is, according to Deligne ([11], (2.77) and (15.6)) the Leray spectral sequence, and by [2], (4.1.1) (i), it degenerates at E_2.
On the other hand, by the regularity of the Gauss-Manin connection, one has

$$\begin{aligned} E^{ab}_{2,an} &= \mathbb{H}^a(S \otimes_{\mathbb{Q}} \mathbb{C}, \Omega^\bullet_{S \otimes_{\mathbb{Q}} \mathbb{C}/\mathbb{C}} \otimes H^b_{DR}(\mathcal{X} \otimes_{\mathbb{Q}} \mathbb{C}/S \otimes_{\mathbb{Q}} \mathbb{C})) \\ &= E_2^{ab} \otimes_{\mathbb{Q}} \mathbb{C} \end{aligned}$$

([1], (6.2) and (7.9)).
This implies that $(E_1^{ab}, d_1) \otimes_{\mathbb{Q}} \mathbb{C}$ degenerates at E_2, and so does (E_1^{ab}, d_1). In particular

$$\begin{aligned} H^j_{DR}(\mathcal{X}/\mathbb{Q}) &= H^0(S, \Omega^\bullet_{S/\mathbb{Q}} \otimes H^j_{DR}(\mathcal{X}/S)) \\ &= \mathrm{Ker}(H^0(S, H^j_{DR}(\mathcal{X}/S)) \longrightarrow H^0(S, \Omega^1_{S/\mathbb{Q}} \otimes H^j_{DR}(X/S))). \end{aligned}$$

This proves the required exactness by base change to $\mathbb{Q}(S)$.

Remark 5 In fact, even if S is not affine, there is a Leray spectral sequence for the de Rham cohomology [7] (3.3), which again degenerates at E_2 by the comparison between the Leray spectral sequences for the Betti and the de Rham cohomologiesi, and the regularity of Gauss-Manin. For more on this, see [8].

Corollary 6 *If α is an absolute Hodge cycle, then it is an absolute de Rham cycle.*

Proof. By [3] (2.5), we know that $\nabla\alpha = 0$, where ∇ is as in (4) for $j = 2i$. Then we apply (4).

Corollary 7 *If α is an absolute de Rham cycle such that $I_\sigma(\alpha) \in H^{2i}_B(X_\sigma, \mathbb{Q})$ for some embedding $\sigma : k \to \mathbb{C}$, then α is an absolute Hodge cycle.*

Proof. In fact, this is [3] (2.6). More precisely, choose S as in the proof of 4 and $\beta \in H^{2i}_{DR}(\mathcal{X}/S)$ restricting to α. The embeddings $\mathbb{Q}(S) \to k \xrightarrow{\sigma} \mathbb{C}$ define a $\mathbb{C}$ valued point of S, which we still denote by σ, such that $\beta(\sigma) \in H^{2i}((\mathcal{X}_{an})_\sigma, \mathbb{Q}) \subset H^{2i}((\mathcal{X}_{an})_\sigma, \mathbb{C})$. The image $\beta(\sigma)$ of β in

$$H^0(S_{an}, R^{2i}f_*\mathbb{C}) = H^{2i}((\mathcal{X}_{an})_\sigma, \mathbb{C})^{\pi_1(S_{an},\sigma)}$$

lies in

$$H^0(S_{an}, R^{2i}f_*\mathbb{Q}) = H^{2i}((\mathcal{X}_{an})_\sigma, \mathbb{Q})^{\pi_1(S_{an},\sigma)}.$$

Therefore $\beta|_{(\mathcal{X}_{an})_s}$ is rational for all s, in particular for those s coming from an embedding $\sigma : k \to \mathbb{C}$.

Remark 8 An advantage, if any, to adopt the language of absolute de Rham cycles consists of dividing the question of wether α is absolute Hodge or not into two steps:

First of all α must be in

$$H^{2i}_{DR}(X_0/k_0) \otimes_{k_0} k_0^{\mathrm{alg}} = \mathrm{Ker} H^{2i}_{DR}(X_0/k_0) \otimes_{k_0} k \longrightarrow \Omega^1_{k/k_0} \otimes_{k_0} H^{2i}_{DR}(X_0/k_0),$$

where k_0^{alg} is the algebraic closure of k_0 in k.
Secondly α must be in

$$\mathrm{Ker} H^{2i}_{DR}(X_0/k_0) \otimes_{k_0} k_0^{\mathrm{alg}} \longrightarrow \Omega^1_{k_0/\mathbb{Q}} \otimes_{k_0} H^{2i}_{DR}(X_0/k_0) \otimes_{k_0} k_0^{\mathrm{alg}}.$$

On the other hand, we have seen that if $\alpha \in F^i H^{2i}_{DR}(X/k)$ is the class of an algebraic cycle, then not only it is an absolute de Rham cycle, but also it is coming from $\mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/\mathbb{Q}})$.

Let $f : \mathcal{X} \to S$, $\beta \in F^i H^j_{DR}(\mathcal{X}/S) = H^0(S, R^j f_* \Omega^{\geq i}_{\mathcal{X}/S})$, such that $\beta \otimes_{\mathbb{Q}(S)} k = \alpha \in$

$F^iH^j_{DR}(X/k)$ as in the proof of 4. Let $f_{\mathbb{C}} : \mathcal{X}_{\mathbb{C}} \to S_{\mathbb{C}}$ be the smooth proper morphism obtained from f by base change $\mathcal{O}_S \otimes_{\mathbb{Q}} \mathbb{C}$, and $\beta_{\mathbb{C}}$ be $\beta \otimes_{\mathbb{Q}} \mathbb{C}$. Let $\overline{f_{\mathbb{C}}} : \overline{\mathcal{X}_{\mathbb{C}}} \to \overline{S_{\mathbb{C}}}$ be a compactification of $f_{\mathbb{C}}$ such that $\Sigma = \overline{S_{\mathbb{C}}} - S_{\mathbb{C}}$, $D = \overline{f_{\mathbb{C}}}^{-1}(\Sigma)$ are normal crossing divisors and $\overline{\mathcal{X}_{\mathbb{C}}}$ is smooth.

Definition 9 A class $\alpha \in F^iH^j_{DR}(X/k)$ is said to be of *moderate growth* if for some $(\beta, \overline{f_{\mathbb{C}}})$ as above, it verifies

$$(*)\ \beta_{\mathbb{C}} \in H^0(\overline{S_{\mathbb{C}}}, R^j\overline{f_{\mathbb{C}}}_*\Omega^{\geq i}_{\overline{\mathcal{X}_{\mathbb{C}}}/\overline{S_{\mathbb{C}}}}(\log D)) \subset H^0(S_{\mathbb{C}}, R^jf_{\mathbb{C}*}\Omega^{\geq i}_{\mathcal{X}_{\mathbb{C}}/S_{\mathbb{C}}})$$

Remark 10 The definition 9 does not depend on the couple $(\beta, \overline{f_{\mathbb{C}}})$ choosen. In fact, take (γ, g) with $g : \mathcal{Y} \to T$, $\mathbb{Q}(T) \subset k$, $\mathcal{Y} \otimes_{\mathbb{Q}(T)} k = X$, $\gamma \otimes_{\mathbb{Q}(T)} k = \alpha$. Then considering in k a function field $\mathbb{Q}(U)$ containing $\mathbb{Q}(S)$ and $\mathbb{Q}(T)$, one has base changes $\sigma : U \to S$, $\tau : U \to T$, $f_U : \mathcal{X}_U = \mathcal{X} \times_S U \to U$, $g_U : \mathcal{Y}_U = \mathcal{Y} \times_T U \to U$, such that there is an isomorphism $\iota : \mathcal{X}_U \to \mathcal{Y}_U$, with $g_U \circ \iota = f_U$, $\iota^*(\gamma \otimes_{\mathcal{O}_T} \mathcal{O}_U) = \beta \otimes_{\mathcal{O}_S} \mathcal{O}_U$, for U small enough. As $\beta_{\mathbb{C}}$ fulfills $(*)$ on $\overline{S_{\mathbb{C}}}$, it fulfills $(*)$ on any blow up $\overline{\sigma_{\mathbb{C}}} : \overline{U_{\mathbb{C}}} \to \overline{S_{\mathbb{C}}}$ such that a commutative diagram exists

$$\begin{array}{ccc} \overline{\mathcal{X}_{U,\mathbb{C}}} & \longrightarrow & \overline{\mathcal{X}_{\mathbb{C}}} \\ {\scriptstyle \overline{f_{U,\mathbb{C}}}}\downarrow & & \downarrow{\scriptstyle \overline{f_{\mathbb{C}}}} \\ \overline{U_{\mathbb{C}}} & \xrightarrow{\overline{\sigma_{\mathbb{C}}}} & \overline{S_{\mathbb{C}}} \end{array}$$

with the properties: $\overline{\sigma_{\mathbb{C}}}^{-1}\Sigma$, $\Delta = \overline{f_{U,\mathbb{C}}}^{-1}\overline{\sigma_{\mathbb{C}}}^{-1}\Sigma$ are normal crossing divisors, $\overline{\mathcal{X}_{U,\mathbb{C}}}$ and $\overline{U_{\mathbb{C}}}$ are smooth. Choose $\overline{U_{\mathbb{C}}}$ such that τ extends to $\overline{\tau_{\mathbb{C}}} : \overline{U_{\mathbb{C}}} \to \overline{T_{\mathbb{C}}}$, with a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{X}_{U,\mathbb{C}}} & \xrightarrow{\overline{\iota_{\mathbb{C}}}} & \overline{\mathcal{Y}_{\mathbb{C}}} \\ {\scriptstyle \overline{f_{U,\mathbb{C}}}}\downarrow & & \downarrow{\scriptstyle \overline{g_{\mathbb{C}}}} \\ \overline{U_{\mathbb{C}}} & \xrightarrow{\overline{\tau_{\mathbb{C}}}} & \overline{T_{\mathbb{C}}} \end{array}$$

with the same properties as above. One has now

$$H^0(\overline{U_{\mathbb{C}}}, R^j\overline{f_{U,\mathbb{C}*}}\Omega^{\geq i}_{\overline{\mathcal{X}_{U,\mathbb{C}}}/\overline{U_{\mathbb{C}}}}(\log \Delta)) = H^0(\overline{T_{\mathbb{C}}}, R^j\overline{g_{\mathbb{C}*}}\Omega^{\geq i}_{\overline{\mathcal{Y}_{\mathbb{C}}}/\overline{T_{\mathbb{C}}}}(\log \overline{g_{\mathbb{C}}}^{-1}(\overline{T_{\mathbb{C}}} - T_{\mathbb{C}}))$$

[6], 4.13.
This implies in particular that classes of moderate growth build a k subvectorspace of $F^iH^j_{DR}(X/k)$.

Notation 11 We denote this subvectorspace by $F^iH^j_{DR}(X,k)^{\log}$, and by $\mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}})^{\log}$ its inverse image in $\mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}})$.

Theorem 12 *The sequence*

$$\mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}})^{\log} \longrightarrow F^iH^j_{DR}(X/k)^{\log} \xrightarrow{\nabla} \Omega^1_{k/\mathbb{Q}} \otimes F^{i-1}H^j_{DR}(X/k)$$

is exact.

Proof. We have to prove that if $\alpha \in \mathrm{Ker}\nabla$, then it lies in the image of $\mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}})$. With the notations as above,

$$\beta_{\mathbb{C}} \in H^0(\overline{S_{\mathbb{C}}}, \Omega^{\bullet}_{\overline{S_{\mathbb{C}}}}(\log \Sigma) \otimes R^j \overline{f_{\mathbb{C}*}} \Omega^{\geq i-\bullet}_{\overline{\mathcal{X}_{\mathbb{C}}}/\overline{S_{\mathbb{C}}}}(\log D)).$$

This group is the E_2^{0j} term of a spectral sequence converging to $\mathbb{H}^j(\overline{\mathcal{X}_{\mathbb{C}}}, \Omega^{\geq i}_{\overline{\mathcal{X}_{\mathbb{C}}}/\overline{S_{\mathbb{C}}}}(\log D))$ and defined as in [7] (3.3) on the complex $\Omega^{\geq i}_{\overline{\mathcal{X}_{\mathbb{C}}}/\overline{S_{\mathbb{C}}}}(\log D)$. One has

$$E_2^{ab} = \mathbb{H}^a(\overline{S_{\mathbb{C}}}, \Omega^{\bullet}_{\overline{S_{\mathbb{C}}}}(\log \Sigma) \otimes R^b \overline{f_{\mathbb{C}*}} \Omega^{\geq i-\bullet}_{\overline{\mathcal{X}_{\mathbb{C}}}/\overline{S_{\mathbb{C}}}}(\log D)).$$

By [6] (0.4) and its analogue in characteristic zero [4] (2.7), E_2^{ab} injects into

$$\mathbb{H}^a(\overline{S_{\mathbb{C}}}, \Omega^{\bullet}_{\overline{S_{\mathbb{C}}}}(\log \Sigma) \otimes R^b \overline{f_{\mathbb{C}*}} \Omega^{\bullet}_{\overline{\mathcal{X}_{\mathbb{C}}}/\overline{S_{\mathbb{C}}}}(\log D)),$$

which is just $H^a(S_{an}, R^b f_{\mathbb{C}*}\mathbb{C})$ by [1] II, §6.
Thus the spectral sequence degenerates at E_2, and $\beta_{\mathbb{C}}$ comes from $\mathbb{H}^j(\overline{\mathcal{X}_{\mathbb{C}}}, \Omega^{\geq i}_{\overline{\mathcal{X}_{\mathbb{C}}}}(\log D))$. In particular $\beta_{\mathbb{C}}$ comes from $\mathbb{H}^j(\mathcal{X}, \Omega^{\geq i}_{\mathcal{X}/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}$ and the image of α in

$$\frac{F^i H^j_{DR}(X/k) \otimes_{\mathbb{Q}} \mathbb{C}}{\mathrm{Im}\ \mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}}) \otimes \mathbb{C}} = \left(\frac{F^i H^j_{DR}(X/k)}{\mathrm{Im}\ \mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}})} \right) \otimes \mathbb{C}$$

vanishes. Therefore α lies in the image of $\mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}})$.

Remark 13 If the transcendence degree of k is ≤ 1, then of course the sequence

$$\mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}}) \longrightarrow F^i H^j_{DR}(X/k) \xrightarrow{\nabla} \Omega^1_{k/\mathbb{Q}} \otimes F^{i-1} H^j_{DR}(X/k)$$

is trivially exact. But if the transcendence degree of k is higher, it is not clear why an absolute Hodge cycle has to be a moderate absolute de Rham cycle.

More generally, one can consider a k subvectorspace V of $H^j_{DR}(X/k)$, such that $I_\sigma(V \otimes_\sigma \mathbb{C})$ is a Hodge substructure of $H^j_{DR}(X_\sigma, \mathbb{C})$. In the light of the above results, one can examine the following questions.

Question 14 Is V stable under the Gauss-Manin connection?

For this, one would like $I_\sigma^{-1}[I_\sigma(V \otimes_\sigma \mathbb{C}) \cap H^j_B(X_\sigma, \mathbb{Q})]$ to lie in V and to be independent of σ.
If so, then V defines a vector bundle $\mathcal{W}$ with a flat connection on S, where S is defined as in 4 such that $V = W \otimes_{\mathbb{Q}(S)} k$, $W \subset H^j_{DR}(X_0/k_0) \otimes_{k_0} \mathbb{Q}(S)$. Then $\mathcal{W}_{an}$ on S_{an} is generated by a local system $\mathcal{F}$.

Question 15 In the above situation, is the monodromy representation associated to $\mathcal{F}$ defined over $\mathbb{Q}$?

Again, one can split up 14 into two parts as in 8. Moreover, the knowledge of 14 does not imply the knowledge of 15.

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