INTERSECTION PAIRING

KAPIL HARI PARANJAPE

We will study curves and line bundles on a smooth projective surface S over a perfect field k.

1. Divisors, Line Bundles and Intersection Pairing

(1) The collection of line bundles L (upto isomorphism) on S forms a group, with $\mathcal{O}_S$ serving as identity, tensor product serving as multiplication, and $L^{-1} = \mathrm{Hom}_S(L, \mathcal{O}_S)$ serving as inverse. This group is denoted as $\mathrm{Pic}(S)$ and called the Picard group of S.

(2) If C is any reduced irreducible curve on S, then the ideal sheaf $I_{C/S}$ is a locally free sheaf of rank 1; the associated line bundle is denoted as $\mathcal{O}_S(-C)$. (This is a consequence of the fact that local rings of S are unique factorisation domains.)

(3) Given a finite collection of curves $\{C_i\}$ and integers n_i, we call the *formal* sum $D = \sum_i n_i C_i$ a divisor in S. We define a line bundle $\mathcal{O}_S(D) = \otimes_i \mathcal{O}_S(-C_i)^{\otimes(-n_i)}$.

(4) The collection of divisors forms an additive group denoted by $\mathrm{Div}(S)$. The above association gives a homomorphism $\mathrm{Div}(S) \to \mathrm{Pic}(S)$.

(5) More generally, given any closed sub-scheme R of S which is locally defined by a single equation, let C_i be the reduced irreducible components of R and net n_i be the value of the equation defining R at the generic point of C_i. Then $I_{R/S} = \otimes_i \mathcal{O}_S(-C_i)^{\otimes n_i}$. By abuse of notation, we can use $\mathcal{O}_S(-R)$ to denote this line bundle and $\mathcal{O}_S(R)$ to denote its dual. We call the associated line bundles and divisors *effective*.

(6) Given a non-zero section s of a line bundle L, we can think of s as defining a homomorphism $L^{-1} \to \mathcal{O}_S$ which identifies L^{-1} with $I_{R/S}$ where $R = Z(s)$ is the locus of zeroes of s. Thus, L is isomorphic to $\mathcal{O}_S(R)$; hence, it is effective.

(7) A line bundle L on S is said to be *very ample* if there is a closed immersion $f : S \hookrightarrow \mathbb{P}^n$ so that L is the pull-back $f^*(\mathcal{O}_{\mathbb{P}^n}(1)$.

(8) A line bundle L is said to be *ample* if $L^{\otimes d}$ is very ample for large enough d.

(9) If L is ample and M is any line bundle (or even coherent sheaf), then $M \otimes L^{\otimes d}$ has a non-zero section for some large enough d. It follows that $M \otimes L^{\otimes d}$ is effective.

(10) Given any surjective homomorphism $M \to N$ of coherent sheaves in S and an ample divisor L on S, one shows that sections of $M \otimes L^d$ surject onto sections of $N \otimes L^d$ for some large enough d.

(11) Combining the above two results one can show that for any line bundle M and an ample line bundle L, the line bundle $M \otimes L^d$ is very ample for all sufficently large d.

(12) If C and D are curves in S meeting at a point P of S, we say that the intersection is *transversal* at P if the local equations f and g respectively of C and D at P generate the maximal ideal of P in the local ring of S at P. Note that this means that f and g are a regular system of parameters for the local ring of S at p and this C and D are also smooth at P.

(13) (Bertini's Theorem) Given any finite set F of (closed) points of S and a finite collection $\{C_i\}$ of (reduced irreducible) curves on S, and a very ample line bundle L on S, there is an affine open set U in the vector space $\Gamma(S, L)$ so that the zero locus $A = Z(s)$ in S of any s in U, satisfies the following conditions:
   (a) A is a smooth irreducible curve in S.
   (b) A does not intersect (i. e. contain any point of) F.
   (c) A is distinct from every curve C_i.
   (d) A meets every curve in C_i transversally at every point lying in $A \cap C_i$.

(14) From what has been said above it follows that for any line bundle M on S we can find smooth curves A and B satisfying the above conditions so that $M = \mathcal{O}_S(A - B)$ and the line bundles $\mathcal{O}_S(A)$ and $\mathcal{O}_S(B)$ are very ample. Moreover, we can also assume that A and B meet transversally (by first choosing one and then the other. In particular, we conclude that $\mathrm{Div}(S) \to \mathrm{Pic}(S)$ is a surjective homomorphism.

(15) For a line bundle (or more generally a coherent sheaf) we define $\chi(L) = h^0(S, L) - h^1(S, L) + h^2(S, L)$ where $h^i(S, L)$ is the rank of the sheaf cohomology $H^i(S, L)$ as a vector space over k.

(16) If C and D are distinct reduced irreducible curves on S, we take the total complex of the tensor product of the complexes $\mathcal{O}_S(-C) \to \mathcal{O}_S$ and $\mathcal{O}_S(-D) \to \mathcal{O}_S$ To obtain a complex

$$0 \to \mathcal{O}_S(-C - D) \to \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-D) \to \mathcal{O}_S$$

Since the local rings are unique factorisation domains where the functions defining C and D and distinct prime elements, we see that this is an exact sequence. Moreover, the cokernel at the last stage is the skyscraper sheaf $\mathcal{O}_{C\cap D}$. By the additivity of χ for terms in an exact sequece it follows that

$$\operatorname{rank}\mathcal{O}_{C\cap D} = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)\chi(\mathcal{O}_S(-D) + \chi(\mathcal{O}_S(-C-D)$$

Here, by abuse of notation, we are writing $\operatorname{rank}\mathcal{O}_{C\cap D}$ for the rank of the global sections of this skyscraper sheaf.

(17) This leads us to define the intersection pairing of two line bundles as follows

$$(L\cdot M) = \chi(\mathcal{O}_S) - \chi(L^{-1}) - \chi(M^{-1}) + \chi(L^{-1}\otimes M^{-1}$$

We also use the notation $(D_1\cdot D_2) = (\mathcal{O}_S(D_1), \mathcal{O}_S(D_2))$ for divisors D_1 and D_2 where there is not possibility of confusion. We note that the formula is clearly symmetric in L and M.

(18) The sheaf $\omega_S = \wedge^2\Omega^1_{S/k}$ is (called) the *canonical* line bundle on S. A divisor K_S such that $\mathcal{O}_S(K_S) = \omega_S$ (in other words, the divisor of zeroes and poles of a meromorphic 2-form on S) is called a *canonical divisor*. Serre duality implies that

$$h^i(S,L) = h^{2-i}(S,\omega_S\otimes L^{-1})$$

for any line bundle L on S. It follows that $\chi(L) = \chi(\omega_S\otimes L^{-1})$.

(19) A simple calculation shows that

$$(L^{-1}\cdot\omega_S^{-1}\otimes L) = 2\chi(\mathcal{O}_S) - 2\chi(L)$$

We deduce the formula (sometimes called the Riemann-Roch formula for surfaces)

$$\chi(L) = \chi(\mathcal{O}_S) - \frac{1}{2}L\cdot(\omega_S\otimes L^{-1})$$

For a divisor D we write this as

$$\chi(D) = \chi(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S) - \frac{1}{2}D\cdot(K_S - D)$$

(20) We can write the Riemann-Roch theorem for curves in the form $\chi(M) - \chi(\mathcal{O}_C) = \deg(M)$ for a line bundle M on a smooth projective curve C. Suppose that C lies on S and that If $M = L_{|C}$ is the restriction to C of a line bundle L om S. We then have the short exact sequence,

$$0\to\mathcal{O}_S(-C)\to\mathcal{O}_S\to\mathcal{O}_C\to 0$$

and its tensor with L,

$$0\to L\otimes\mathcal{O}_S(-D)\to L\to L_{|C}\to 0$$

By the additivity of χ in exact sequences, we have

$$\deg(L_{|C}) = \chi(L) - \chi(L\otimes\mathcal{O}_S(-D)) - \chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(-C))$$

In other words, $\deg(L_{|C}) = -(L^{-1}\cdot\mathcal{O}_S(C))$. Replacing L by L^{-1}, we obtain $(L\cdot\mathcal{O}_S(C)) = \deg(L_{|C})$.

(21) We now define the symbol (L_1, L_2, L_3) as

$$(L_1, L_2, L_3) = L_1\cdot L_3 + L_2\cdot L_3 - (L_1\otimes L_2)\cdot L_3$$

If L_3 is $\mathcal{O}_S(C)$ for a smooth curve, then by the additivity of the degree of line bundles on a curve, we see that $(L_1, L_2, \mathcal{O}_S(C)) = 0$. We obtain the following symmetric expression for (L_1, L_2, L_3) by expanding all the terms

$$\chi(\mathcal{O}_S) - \chi(L_1^{-1}) - \chi(L_2^{-1} - \chi(L_3^{-1}) + \chi(L_1^{-1}\otimes L_3^{-1}) + \chi(L_2^{-1}\otimes L_3^{-1}) + \chi(L_1^{-1}\otimes L_2^{-1}) - \chi(L_1^{-1}\otimes L_2^{-1}\otimes L_3^{-1})$$

It follows that for any two line bundles L and M and any smooth projective curve C on S,

$$(L\cdot M) = (L\otimes\mathcal{O}_S(C)\cdot M) - \deg(M_{|C})$$

Now, we write $L = \mathcal{O}_S(A-B)$ for suitable smooth projective curves A and B on S, and take $C = B$ o obtain

$$(\mathcal{O}_S(A-B)\cdot M) = \deg(M_{|A}) - \deg(M_{|B})$$

The right hand side is additive in M and the line bundle L is arbitrary. It follows that $(L_1, L_2, L_3) = 0$ for all line bundles L_i; equivalently, the intersection pairing is bi-additive.

In summary, we have shown that intersection pairing of divisors is a symmetric bi-additive pairing.

2. $\mathbb{Q}$-DIVISORS, NEF, AMPLE, BIG, ETC.

(1) We will now tensor the above groups with the rational numbers $\mathbb{Q}$ making them all vector spaces over $\mathbb{Q}$. We thus have a surjective homomorphism $\mathrm{Div}(S)_{\mathbb{Q}} \to \mathrm{Pic}(S)_{\mathbb{Q}}$ and a pairing $(L \cdot M)$ giving a rational number for every pair L and M of elements of $\mathrm{Pic}(S)_{\mathbb{Q}}$.

(2) We say that a line bundle L on S *numerically equivalent* to 0 if $L \cdot M = 0$ for every line bundle M on S. Note that this is equivalent to the assertion that $\deg(L_{|C} = 0$ for every smooth irreducible curve C on S. Such line bundles form a subgroup of $\mathrm{Pic}(S)$.

(3) We now define the Néron-Severi group (vector space!) $\mathrm{NS}(S)_{\mathbb{Q}}$ to be the quotient of $\mathrm{Pic}(S)_{\mathbb{Q}}$ by the subspace generated by line bundles that are numerically equivalent to 0. We use $[L]$ to denote the class in $\mathrm{NS}(S)_{\mathbb{Q}}$ of a line bundle L on S. By abuse of notation, we use $[C]$ to denote the class $[\mathcal{O}_S(C)]$. We note that there is a non-degenerate pairing on $\mathrm{NS}(S)_{\mathbb{Q}}$ induced by the pairing introduced earlier.

(4) It is known that $\mathrm{NS}(S)_{\mathbb{Q}}$ is a finite dimensional vector space.

(5) Let $\mathrm{Div}(S)^{\geq 0}$ denote that subset consisting of non-negative integer combinations of reduced irreducible curves on S. Given A and B in it, and non-negative integers a and b, we see that $aA + bB$ also lies in it.

(6) The *cone of effective divisors* in $\mathrm{NS}(S)_{\mathbb{Q}}$ is the cone of non-negative rational linear combinations of classes $[C]$ of reduced irreducible curves.

(7) The *cone of nef divisors* is the *dual cone* of the cone of effective divisors. In other words, we say that D is *nef* if $D \cdot C \geq 0$ for every reduced irreducible curve.

(8) If C and D are distinct reduced irreducible curves on S, we have seen above that $C\dot{D}$ is the length of $\mathcal{O}_{C\cap D}$; in particular, it is non-negative. It follows that C is nef *unless* $C\dot{C} < 0$. More generally, if $F = \sum_i a_i[C_i]$ is a non-negative linear combination in the cone of effective curves, then F is nef *unless* $F \cdot C_i < 0$ for some i.

(9) Suppose that C and C' are distinct curves so that $[C] = [C']$. We see that $[C] \cdot [C] = [C] \cdot [C'] \geq 0$. It follows that $[C]$ is nef. In other words, a curve that "moves" is automatically nef.

(10) Suppose that A is a very ample line bundle associated with a closed embedding $S \hookrightarrow \mathbb{P}^n$. We have $A \cdot C = \deg(A_{|C}) > 0$ for all curves C. It follows that any ample divisor is nef.

(11) If L is any line bundle and E an effective divisor, the $A \cdot L(-nE) \to -\infty$ as $n \to \infty$. It follows that $L(-nE)$ cannot be effective for large n for any effective divisor E. In particular, we note that D and $-D$ cannot both be (sub)-multiples of effective divisors.

(12) If $D \cdot D > 0$ then, by the Riemann-Roch for surfaces given above,

$$h^0(nD) + h^0(K_S - nD) \geq \chi(nD) = \chi(\mathcal{O}_S) - (D \cdot K_S) + n^2(D \cdot D) \to \infty \text{ as } n \to \infty$$

It follows, for large n, that either nD or $K_S - nD$ is effective. In fact, this leads to a *dichotomy*:

(a) Suppose that $h^0(nD) = 0$ for all large n. Let m be such that $K_S - mD$ is an effective divisor E. Then $h^0(E - nD)$ grows like cn^2 for a positive constant c. However, $h^0(\mathcal{O}_S(E - nD) \otimes \mathcal{O}_E)$ grows at most like dn (since E is a curve). It follows that $h^0(-nD)$ grows like en^2 for some positive constant e.

(b) On the other hand, if mD is effective for some m, then as seen above $K_S - nmD$ cannot be effective for larger) for all large n. Thus, $h^0(nmD)$ grows like cn^2 for some positive constant c.

In other words, ether there is a multiple D_1 of D so that $h^0(nD_1)$ grows like a positive multiple of n^2 or $h^0(-nD)$ grows like a positive multiple of n^2.

(13) A divisor D is called *big* if $h^0(nD)$ grows like a positive multiple of n^2. What we have proved above is that if $D \cdot D > 0$, then there is a multiple D_1 of D (positive or negative) which is big. Note that some multiple of a big divisor is effective and so, for any ample line bundle A we have $A \cdot D \neq 0$.

(14) We deduce the Hodge Index theorem: If $D \cdot A = 0$ for some ample divisor A, then $D \cdot D < 0$.