INTERSECTION PAIRING

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We will study curves and line bundles on a smooth projective surface S over a perfect field k.

1. Negativity of Intersection Forms on Fibre Components

(1) Suppose that V is a $\mathbb{Q}$-vector space with a symmetric bi-linear form (indicated by $(\cdot)$). Given $e_i \in V$ a finite collection of elements so that $e_i \cdot e_j \geq 0$ for $i \neq j$. Suppose $z = \sum_i e_i$ is such that $z \cdot e_i \leq 0$.

(2) Putting $c = \sum_i c_i e_i$, we see that

$$c \cdot c = \sum_i c_i^2 e_i \cdot e_i + 2\sum_{i<j} 2c_i c_j e_i \cdot e_j$$

$$\leq \sum_i c_i^2 e_i \cdot e_i + \sum_{i<j}(c_i^2 + c_j^2) e_i \cdot e_j$$

$$= \sum_i c_i^2 e_i \cdot e_i + \sum_{i<j} c_i^2 e_i \cdot e_j$$

$$\sum_{i>j} c_i^2 e_i \cdot e_j$$

$$= \sum_i \sum_j c_i^2 e_i \cdot e_j = \sum_i c_i^2 z \cdot e_i \leq 0$$

It follows that $c\dot{c} \leq 0$ with equality only if $2c_i c_j = c_i^2 + c_j^2$ (equivalently, $c_i = c_j$) whenever $e_i \cdot e_j > 0$ *and* $z \cdot e_i = 0$ for all i such that $c_i \neq 0$.

(3) Replacing e_i by a positive multiple of e_i, we see that the same result holds with z replaced by a positive linear combination of the e_i's.

(4) We now apply this to the case where e_i are the classes of components of the fibre over a (smooth) point b of a morphism $f : S \to C$ where S is a smooth surface and C is a curve. Since the fibre divisor z is a positive linear combination of an effective divisor supported on the point. On the other hand, any divisor supported on b is linearly equivalent to a divisor whose support excludes b. Hence $z \cdot e_i = 0$ and the above calculations apply to show that *the intersection pairing on the subspace generated by the fibre components of a map from a smooth surface to a curve is negative semi-definite*.

(5) If we moreover assume that the fibre is connected then for all i there is a j so that $e_i \cdot e_j \neq 0$. In this case, the pairing is negative definite.

(6) Suppose that there is a dominant morphism $f : S \to S'$ for a surface S' so that the image $f(E_i) = p$ of the curves E_i in S is (the same) point p for all i. Let D_i be curves in S so that $E_i \cdot D_i > 0$. Let g be a regular function near p on S' that vanishes on the images of D_i. We write the divisor of zeros of g on S as $D'' = \sum_i a_i E_i + D$ where D consists of all components other than E_i; we put $Z = \sum_i a_i E_i$. The divisor D'' is linearly equivalent to the divisor D' of poles of g on S and the image of D' in S' avoids E_i. It follows that $E_i \cdot D'' = E_i \cdot D' = 0$. Since $E_i \cdot D > 0$, it follows that $Z \cdot E_i < 0$. The above linear algebra then implies that *the intersection pairing on the subspace generated by the components of the fibres of a map from a smooth surface to another surfaces is negative definite.*