INTRODUCTION TO THE ALGEBRAIC SURFACES WORKSHOP

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1. The Beginning

A number of different strands combined to become Algebraic Geometry as we know it today.

- The study of Riemann surfaces and algebraic curves. This was seen more algebraically as the study of Dedekind domains and a field of transcendence degree one. This study made contact with the study of algebraic number theory as initiated by Kummer and Kronecker.
- The study of Elliptic functions with Eisenstein series, the Weierstrass $\wp$-function, theta functions and so on led to the study of Abelian varieties and related varieties.
- Synthetic projective geometry went on the study of enumerative problems and the introduction of Grassmanians and other homogeneous varieties. It also got merged with the study of affine algebraic groups.
- The study of commutative algebra and homological algebra introduced a systematic method to prove all local (and some global) properties of algebraic varieties.
- The study of algebraic and differential topology. The relation between topological properties and differential invariants exemplified by (Say) the Gauss-Bonnet theorem. The study of Lie groups, especially matrix groups and their classifying spaces also merges in.

Most of you have seen some basics of the above concepts and have probably studied algebraic curves in some detail. However, none of the above topics by themselves exhibit the full glory of what algebraic geometry has evolved into. (For example, one can get by with only a little commutative algebra (Dedekind domains) when studying algebraic curves.)

It is only when one starts studying Algebraic Surfaces that all of the above strands come together in a way that none of the strands can be extricated from the other.

2. Some primary questions

An algebraic surface can be defined as an irreducible, reduced algebraic scheme X of dimension 2 over a field k. Some people may instead ask for a 2-dimensional complex manifold and some other (not fully equivalent!) variants are possible as well.

With some basic knowledge of algebraic geometry one can immediately ask many questions. For example,

- Is every field of transcendence degree 2 associated with an algebraic surface? Is such a surface (if projective) unique? Can there be a non-quasi-projective surface associated with such a field?
- If a field of transcendence degree 2 is contained in the rational function field of two variables, is such a field automatically isomorphic to the rational function field of two variables? (Luroth's problem)
- Does the topology of an algebraic surface determine its "type" like in the case of curves? In other words, is there only one (connected) component for the moduli of algebraic structures for a fixed topological type? Are there surface "types" that cannot be found as hypersurfaces in projective three space?
- What are the possible homotopy types of algebraic surfaces? (Recall that the homotopy types of curves are rather limited. In particular, is there any restriction on what groups can be the fundamental groups of surfaces?
- The second homology of a surface will have a quadratic form (due to Poincare duality). What kinds of quadratic forms are possible. Are all classes in the second homology representable by algebraic curves on the surface?
- Is there any difference between surfaces in characteristic p and those in characteristic 0? Recall that the "types" of curves are the same in all characteristics.
- Is there a classification of surfaces similar to that for curves?
- Is there a theory of linear systems on surfaces analogous to the that for curves. Are there notions line Weierstrass points etc.? What is the minimal dimension of a projective space in which one can embed a projective algebraic surface.

During these two weeks we will attempt to look at some of these questions and find suitable answers.

3. Techniques

We will use a number of "standard" algebraic tools during the lectures that follow. These tools will be quickly recalled as we go along. In addition, there are a number of techniques that we can utilise to study surfaces.

Geometric: We can realise a projective algebraic surface in a number of different ways: as a covering of the projective plane, as a family of curves parametrised by a curve, as a hypersurface in projective three space (or another three dimensional homogeneous space), as a variety dominated by a product of two curves.

Topological: We can study the topology of a surface by studying linear systems of curves on it, or by studying the properties of intersections of such curves. We can also study divisors in various linear (and non-linear) systems of curves on the surface. One important topological tool is that of a Lefschetz pencil which generalises the notion of a curve covering the projective line only simple ramification points.

Differential: In addition to 1-forms, we can also study holomorphic 2-forms on a surface. There are forms of various kinds giving rise to an algebraic version of the de Rham Theorem. The important complex algebraic technique, Hodge theory plays a much more significant role in the study of surfaces that in does in the theory of curves.

Birational: While smooth curves are determined by their function field, there are a number of smooth surfaces with the same function field. The study of how to go from one such surface from another (blowing-up and blowing-down) as well as the study of numerical invariants that are unchanged by such operations is a useful technique.

4. Plan

It is not possible to study surfaces in the linear order that one studies curves or even abelian varieties. Thus, the approach that we will follow is best explained by analogy with a music class! None of the lecturers will attempt to give a "complete" proof of a theorem (just as a music teacher does not give a concert during a class!). Instead, we will tell you about the important things to watch for and the main results to attain some grasp of (the "vaadi" and the "samvaadi"!). We will have tutorials where (I hope!) some of you will try some problems so that we can point out where you are taking a convoluted path and where your short-cut will not work and why.

Note that strict linearity is not going to be maintained. So some lectures will use concepts that will be explained in later lectures; you will just have to suspend disbelief and move on. Each lecture will introduce one or more concepts and explain them through examples and through key results; they will occasionally highlight some key points in the proofs of these result, but most often, especially in later lectures, proofs will be skipped entirely or left for discussion in the tutorial sessions. This is slightly different from the usual style of lecturing in mathematics. Let's see how it works out!

5. Mathematical introduction

With that (rather too verbose for some!) introduction out of the way, let us get to some mathematics.

An affine algebraic surface cab be defined as the "locus of zeroes" of a collection $f_1(x_1,\ldots,x_p)$,…,$f_q(x_1,\ldots,x_p)$ of polynomial equations over a field k, so that the ring

$$R = \frac{k[x_1,\ldots,x_p]}{< f_1,\ldots,f_q >}$$

has the following properties:

(1) R is a domain.

(2) R is two dimensional. This means that there is pair of elements u_1, u_2 in R so that the sub-algebra $k[u_1,u_2]$ generated by them is a polynomial algebra (i. e. they are algebraically independent) and the inclusion $k[u_1,u_2] \to R$ is an integral (and separable) extension.

(3) Very often we will assume that R is smooth over k, which means for each maximal ideal m of R we can choose u_1, u_2 so that, the extension $k[u_1,u_2] \to R$ has the following properties:

   (a) The ideal in R_m generated by (u_1,u_2) is the maximal ideal mR_m.

   (b) The resulting field extension $k \to R/m$ is a separable extension.

   When these two conditions are satisfied we say that $k[u_1,u_2] \to R_m$ is étale.

Note that the field K of quotients of R is a finite separable extension of $k(u_1, u_2)$. Conversely, given such an extension, we can take R to be the integral closure of $k[u_1, u_2]$ in K. One proves (exercise!) that R is a finitely generated k-algebra and (by Hilbert's basis theorem) that the kernel ideal is generated by finitely many polynomials.

The remaining question of interest is to "compactify" this surface. For this, we pick a filtering $R_0 = k \subset R_1 \subset \dots$ of R by finite dimensional k vector spaces R_n so that $R_n \cdot R_m \subset R_{m+n}$. For example, we can let R_n by the image of polynomials of degree at most n in the variables $x_1, \dots, x_q$. We then form the ring $S = \oplus_n R_n$ and note that it is a graded ring. If we have chosen well(!), S is generated by R_1, and so it is a quotient of the polynomial ring as a graded ring. Hence,

$$S = \frac{k[X_0, X_1, \dots, X_p]}{< F_1, \dots, F_s >}$$

for some *homogeneous* polynomials $F_1, \dots, F_s$. You can ask yourself (exercise!) what the generators X_i and whether you can determine the polynomials F_i. The locus of zeroes of $F_1, \dots, F_s$ is a projective algebraic surface containing our original surface as an affine open sub-variety.

In general, one would like to think of an algebraic surface as "made up of affine surfaces by patching". The question which remains is what kind of patching we "permit". In the theory of schemes, we use the Zariski topology for patching. If you are a complex geometry person you may want to allow more general constructions. For example, we may allow étale patching as was done by Moishezon. This sometimes gives us compact complex surfaces that are not algebraic (and yet have a function field of transcendence degree two). However, in the case of surfaces, such surfaces *have to be* singular. (In three or more dimensions there are even smooth examples.) This already is a bit of a contrast with the theory of curves where there are no such examples.

The "simplest" algebraic surfaces are surely $\mathbb{A}^2$, the affine plane and $\mathbb{P}^2$, the projective plane. The fundamental formula for (projective) plane curves is Bezout's theorem which says that a curve of degree m meets a curve of degree n in mn points *if counted properly*. In general, calculating the intersection of *distinct* curves in a surface follows the same approach. Things become interesting when we want to calculate the (virtual) intersection number of a curve with itself. In order to preserve linearity of such intersections, one arrives at a canonical intersection number (at least for smooth surfaces)—in some cases the number can even be negative; which may not be surprising from a topological perspective.

It turns out that each curve C in a smooth surface X (or in the smooth locus of a singular surface) gives rise to a line bundle $\mathcal{O}_X(C)$ on X and hence a class $[C] = c_1(\mathcal{O}_X(C))$ (first Chern class of the line bundle) in the second cohomology $H^2(X)$ of the surface. We can therefore calculate the cap of this class with any other class in $H^2(X)$. For the class $[D]$ of another curve in X, it turns out that $[C] \cap [D] = (C \cdot D)[p]$ where $C \cdot D$ is the intersection number calculated algebro-geometrically and $[p]$ is the class in $H^4(X)$ of a (any) smooth point p on X.

The above interplay between curves on a surface, line bundles on the surface and the associated homology classes is a very interesting and important aspect of the study of surfaces.

We have already mentioned the module of differentials $\Omega^1_{R/k}$ for a ring R. This patches up to give a coherent sheaf $\Omega^1_{X/k}$ on a scheme X over k. By taking exterior powers we can form the differential graded algebra $\Omega^{\cdot}_{X/k}$ just as we do for manifolds. When X is smooth over k, this is exact for the *completed* localisation, but *not* in general for in affine open sets. Grothendieck, generalised the classical theory of differential forms of the first and third kind and pointed out that this DGA can none-the-less be used to calculate the topological structure of X. This result and Serre duality underline the importance of studying 1-forms and 2-forms on a surface.

The projective plane is by no means the only surface whose field of rational functions is isomorphic to $k(x, y)$. Take a rational normal curve C_a in $\mathbb{P}^a$ and a rational normal curve C_b in $\mathbb{P}^b$ and put $\mathbb{P}^a$ and $\mathbb{P}^b$ in a disjoint fashion as linear subspaces in $\mathbb{P}^{a+b+1}$; moreover, pick an isomorphism between C_a and C_b. The union of the lines that join pairs of corresponding points on C_a and C_b gives a surface $F_{|a-b|}$, called a Hirzebruch surface. By construction it is "ruled"; in that it is a union of lines. More generally, given any map of a curve C to the Grassmannian $G(1, N)$ of lines in $\mathbb{P}^N$ gives rise to a ruled surface in $\mathbb{P}^N$. This is an important class of surfaces that we will study. How about families of rational curves of higher degree? This is question behind Tsen's theorem.

One natural generalisation of the theory of Elliptic curves is the study of compact complex tori of dimension two. When such a surface has non-trivial rational functions (which is not always!), then one can show that it is a projective variety and that the group structure is algebraically defined. This leads to the study of Abelian Surfaces which has many aspects that are more detailed and intriguing than the study of general Abelian varieties of higher dimensions.

In any detailed study of projective curves or of Riemann surfaces, we are introduced to the Abelian variety called the Jacobian $J(C)$ of the curve C. This is an *algebraic* form of the group of line bundles of degree 0 on the curve; it is also the "initial object" in the category of abelian varieties admitting a map from C. In the case of surfaces, these two varieties can be distinct and are called the Picard and Albanese varieties of the surface. We will show some key ideas behind the construction of these important invariants of a surface.

A different kind of generalisation of the notion of an elliptic curve is the notion of a K3 surface. This is a simply connected surface which has a global nowhere vanishing 2-form. It was proved by Kodaira that all such surfaces have the same topological type; however the moduli space is not as simple as one might think as there are K3 surfaces which are not algebraic! The existence of K3 surfaces is what clearly indicates that the classification of surfaces is much more complicated than that of curves — we have two generalisations of elliptic curves!

Another aspect of the study of surfaces is that singularities of surfaces have a lot more topological information that just the bunching together of points which happens on a curve. This leads to the fascinating study of surface singularities; we begin with rational double points, which is a kind of singularity that does not exist in dimension one!

There are a few other topics that will be touched upon during the second week that it is difficult to introduce at this point. All in all, we are doing our best to throw as much of surface theory that we (the speakers) have some handle on. We hope you will catch some of these throws and get infected with the enjoyment of this fascinating subject.