An Invitation to Science and Mathematics
The Institute of Mathematical sciences

An application of Trigonometry - Heights and Distances

1 Revision :

Let us first recall the things we have studied about trigonometry :

- **Trigonometry** is the study of measurements of triangles.
- An **angle** is a portion of the 2-dimensional plane which resides between two different directed line segments. The starting position of the angle is known as the initial side and the ending position of the angle is known as the terminal side. The point from which both of the directed line segments originate is known as the **vertex** of the angle.

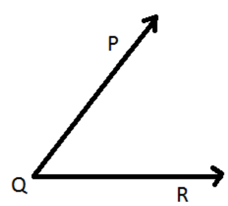


Figure 1: Here Q is the vertex, QR is the initial side and QP is the terminal side. Angle is denoted by $\angle PQR$

- A **right-angled triangle** is a special type of triangle where one of its interior angle is 90^o. Side opposite to 90^o is called the **Hypotenuse**.
- **Pythagoras Theorem :** In a right-angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the other two sides.

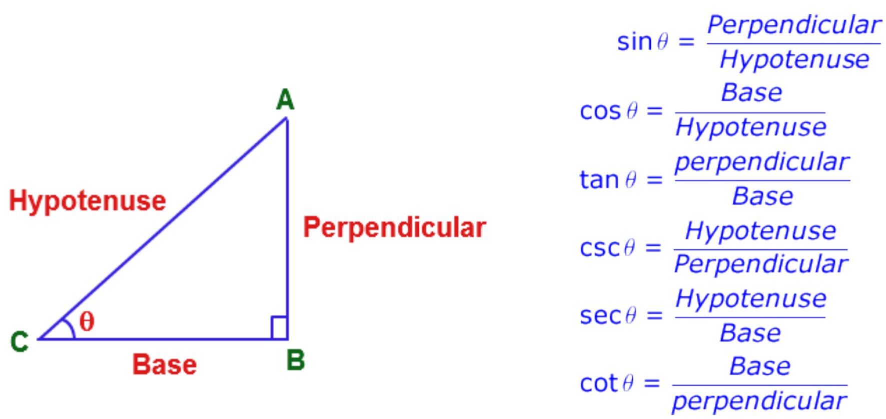


Figure 2: Trigonometric ratios

Angles in Degrees	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Not defined
csc	Not defined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
sec	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Not defined
cot	Not defined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Figure 3: Values of ratios for standard angles

2 Application :

Now let us see some real life applications of Trigonometry. Trignometry is interesting and has many useful applications in the field of astronomy, geography etc. One of the useful applications of trignometry is to calculate the height of a tower or a peak or the distance of a ship sailing in the sea. we do not need measuring scales. All we need to know is the angle of elevation or angle of depression.

Theodolite is an instrument which is used in measuring the angle between an object and the eye of the observer. A theodolite consists of two graduated wheels placed at right angles to each other and a telescope. The wheels are used for the measurement of horizontal and vertical angles. The angle to the desired point is measured by positioning the telescope towards that point. The angle can be read on the telescopic scale.

Figure 4: Theodolite

3 Definitions :

Let us define a few terminologies so that we can understand and implement the concept of Trigonometry to find the heights and distances.

- **Line of sight** is a straight line from our eye to the object.
- If an object is below the horizontal line from the eye, we have to lower our head to view the object. In this process our eyes moves through an angle. This angle is called the **angle of depression**. The angle of depression of an object viewed is the angle formed by the line of sight with the horizontal line, when the object is below the horizontal line.
- Similarly, if an object is above the horizontal line from our eyes we have to raise our head to view the object. In this process our eyes move through an angle formed by the line of sight and horizontal line which is called the **angle of elevation**.

Few things to remember before we start solving problems :

- Read the statements of the question carefully.
- Construct a diagram for the problem for better understanding and visualization of the problem.

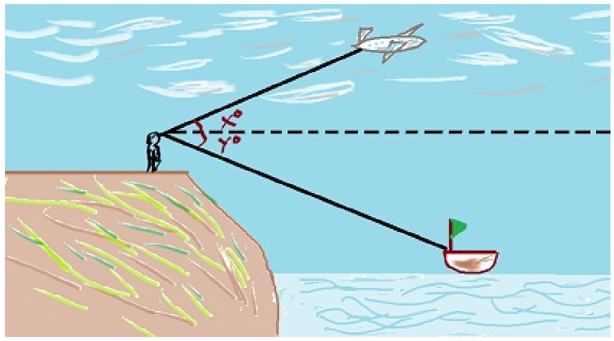


Figure 5: x^o is the angle of elevation and y^o is the angle of depression

- Identify the trigonometrical ratio that will be useful for solving the problem.
- Real world problems will not have standard angles always. So use the below table whenever and wherever necessary.

TABLE OF TRIGONOMETRIC RATIOS

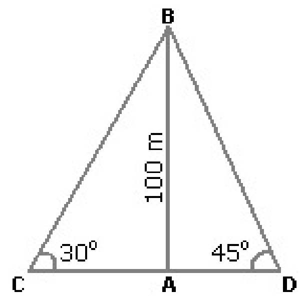
θ	*sin*(θ)	*cos*(θ)	*tan*(θ)	θ	*sin*(θ)	*cos*(θ)	*tan*(θ)
0	0.000	1.000	0.000	45	0.707	0.707	1.000
1	0.017	1.000	0.017	46	0.719	0.695	1.036
2	0.035	0.999	0.035	47	0.731	0.682	1.072
3	0.052	0.999	0.052	48	0.743	0.669	1.111
4	0.070	0.998	0.070	49	0.755	0.656	1.150
5	0.087	0.996	0.087	50	0.766	0.643	1.192
6	0.105	0.995	0.105	51	0.777	0.629	1.235
7	0.122	0.993	0.123	52	0.788	0.616	1.280
8	0.139	0.990	0.141	53	0.799	0.602	1.327
9	0.156	0.988	0.158	54	0.809	0.588	1.376
10	0.174	0.985	0.176	55	0.819	0.574	1.428
11	0.191	0.982	0.194	56	0.829	0.559	1.483
12	0.208	0.978	0.213	57	0.839	0.545	1.540
13	0.225	0.974	0.231	58	0.848	0.530	1.600
14	0.242	0.970	0.249	59	0.857	0.515	1.664
15	0.259	0.966	0.268	60	0.866	0.500	1.732
16	0.276	0.961	0.287	61	0.875	0.485	1.804
17	0.292	0.956	0.306	62	0.883	0.469	1.881
18	0.309	0.951	0.325	63	0.891	0.454	1.963
19	0.326	0.946	0.344	64	0.899	0.438	2.050
20	0.342	0.940	0.364	65	0.906	0.423	2.145
21	0.358	0.934	0.384	66	0.914	0.407	2.246
22	0.375	0.927	0.404	67	0.921	0.391	2.356
23	0.391	0.921	0.424	68	0.927	0.375	2.475
24	0.407	0.914	0.445	69	0.934	0.358	2.605
25	0.423	0.906	0.466	70	0.940	0.342	2.747
26	0.438	0.899	0.488	71	0.946	0.326	2.904
27	0.454	0.891	0.510	72	0.951	0.309	3.078
28	0.469	0.883	0.532	73	0.956	0.292	3.271
29	0.485	0.875	0.554	74	0.961	0.276	3.487
30	0.500	0.866	0.577	75	0.966	0.259	3.732
31	0.515	0.857	0.601	76	0.970	0.242	4.011
32	0.530	0.848	0.625	77	0.974	0.225	4.331
33	0.545	0.839	0.649	78	0.978	0.208	4.705
34	0.559	0.829	0.675	79	0.982	0.191	5.145
35	0.574	0.819	0.700	80	0.985	0.174	5.671
36	0.588	0.809	0.727	81	0.988	0.156	6.314
37	0.602	0.799	0.754	82	0.990	0.139	7.115
38	0.616	0.788	0.781	83	0.993	0.122	8.144
39	0.629	0.777	0.810	84	0.995	0.105	9.514
40	0.643	0.766	0.839	85	0.996	0.087	11.430
41	0.656	0.755	0.869	86	0.998	0.070	14.301
42	0.669	0.743	0.900	87	0.999	0.052	19.081
43	0.682	0.731	0.933	88	0.999	0.035	28.636
44	0.695	0.719	0.966	89	1.000	0.017	57.290
45	0.707	0.707	1.000	90	1.000	0.000	

Figure 6: Ratio table

4 Solved Problems

1. Two ships are sailing in the sea on the two sides of a lighthouse. The angle of elevation of the top of the lighthouse is observed from the ships are 30^o and 45^o respectively. If the lighthouse is $100m$ high. What is the distance between the two ships?

   Solution : Let AB be the lighthouse and C and D be the positions of the ships.

   Here, we need to find CD

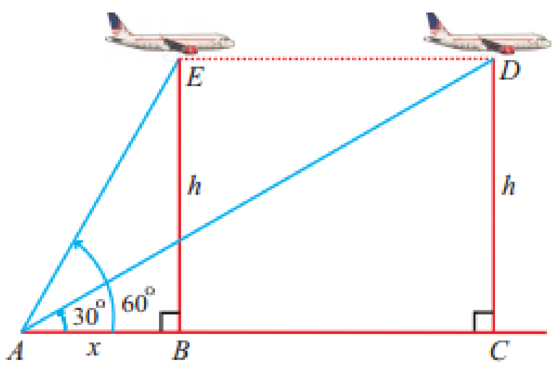
$$AB = 100m, \angle ACB = 30^o and \angle ADB = 45^o$$
$$\frac{AB}{AC} = \tan 30^o = \frac{1}{\sqrt{3}} \implies AC = AB \times \sqrt{3} = 100\sqrt{3}m$$
$$\frac{AB}{AD} = \tan 45^o = 1 \implies AB = AD = 100m$$

   Now,

$$CD = (AC + AD)$$
$$= (100\sqrt{3} + 100)m$$
$$= 100(\sqrt{3} + 1)m$$
$$= (100 \times 2.73)m$$
$$= 273m$$

2. The angle of elevation of an aeroplane from a point A on the ground is 60^o . After a flight of 15 seconds horizontally, the angle of elevation changes to 30^o . If the aeroplane is flying at a speed of $200m/s$, then find the constant height at which the aeroplane is flying.

Solution : Let A be the point of observation.
Let E and D be positions of the aeroplane initially and after 15 seconds respectively. Let BE and CD denote the constant height at which the aeroplane is flying. Given that $\angle DAC = 30^o$ and $\angle EAB = 60^o$.



Let $BE = CD = h$ metres.
Let $AB = x$ metres.
The distance covered in 15 seconds, $ED = 200 \times 15 = 3000m$
Thus, $BC = 3000m$
In right-angled $\triangle ADC$,
$\tan 30^o = \frac{CD}{AC}$
$\implies CD = AC \times \tan 30^o$
Thus,

$$h = (x + 3000) \times \frac{1}{\sqrt{3}} \qquad (1)$$

In right-angled $\triangle AEB$,
$\tan 60^o = \frac{BE}{AB}$
$\implies BE = AB \times \tan 60^o$
Thus,

$$h = \sqrt{3} \times x \qquad (2)$$

From (1) and (2), we have
$\sqrt{3} \times x = (x + 3000) \times \frac{1}{\sqrt{3}}$

$\implies 3x = x + 3000$
$\implies x = 1500m$
Thus, from (2), $h = 1500\sqrt{3}$ m

5 Practice problems

1. The angles of elevation of an artificial earth satellite is measured from two earth stations, situated on the same side of the satellite, are found to be 30^0 and 60^0. If the distance between the earth stations is $4000km$, find the distance between the satellite and earth. ($\sqrt{3} = 1.732$)

2. From the top and foot of a 40 m high tower, the angles of elevation of the top of a lighthouse are found to be 30^o and 60^o respectively. Find the height of the lighthouse. Also find the distance of the top of the lighthouse from the foot of the tower.

3. A vertical tree is broken by the wind. The top of the tree touches the ground and makes an angle 30^o with it. If the top of the tree touches the ground 30 m away from its foot, then find the actual height of the tree.

4. A simple pendulum of length 40 cm subtends 60^o at the vertex in one full oscillation. What will be the shortest distance between the initial position and the final position of the bob? (between the extreme ends)

5. Two crows A and B are sitting at a height of 15 m and 10 m in two different trees vertically opposite to each other. They view a vadai (an eatable) on the ground at an angle of depression 45c and 60crespectively. They start at the same time and fly at the same speed along the shortest path to pick up the vadai. Which bird will succeed in it?