1 Congruences

1. Solve the following congruences:

$$3x \equiv 2 \pmod 5, \quad 2x \equiv 8 \pmod{11}, \quad 12x \equiv 8 \pmod{16}, \quad 18x \equiv -12 \pmod{17}$$

2. Let x be any integer. Prove that:

   (a) $x^2 \equiv 0,\ 1 \text{ or } 4 \pmod 5$.

   (b) $x^3 \equiv 0,\ 1 \text{ or } 8 \pmod 9$.

   (c) $x^4 \equiv 0,\ 1,\ 3 \text{ or } 9 \pmod{13}$.

   (d) $x^2 \equiv 0,\ 1,\ 2 \text{ or } \ 4 \pmod 7$.

   (e) $x^5 \equiv 0,\ 1 \text{ or } 10 \pmod{11}$.

   (f) $x^5 \equiv x \pmod 5$.

3. Prove that the last digit of a perfect square is either $0,\ 1,\ 4,\ 5,\ 6$ or 9.

4. Let a and b be integers such that $7 \mid a^2 + b^2$. Prove that $7 \mid a$ and $7 \mid b$.

5. Suppose that $x^2 \equiv 1 \pmod 8$. Prove that $x^2 \equiv 1 \pmod{16}$.

6. Let m, a, b and c be integers such that $(c, m) = d$. If $ac \equiv bc \pmod m$, show that $a \equiv b \pmod{\frac{m}{d}}$

*7. Integers (x, y, z) are said to form a *Pythagorean Triplet* if $x^2 + y^2 = z^2$, or in other words if x, y and z form the sides of a right angled triangle. For example, $(3, 4, 5)$ is a pythagorean triplet since $3^2 + 4^2 = 5^2$. Other pythagorean triplets are $(5, 12, 13)$. $(7, 24, 25)$ etc. Prove that if (x, y, z) is a Pythagorean triplet, then $60 \mid xyz$.

*8. Prove that for any choice of integers a and b, one has $30 \mid a^5 b - ab^5$.

*9. Prove that $97 \mid 2^{48} - 1$ and that $73 \mid 8^{36} - 1$.