1 Induction

Prove the following statements for all positive integers n using induction.

1. $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.
2. $2^n \geq 1 + n$
3. $n(n+1)(n+2)$ is divisible by 6.
4. Let p be a prime. We know that if $p \mid ab$, then $p \mid a$ or $p \mid b$. Show using induction on n that if $p \mid a_1 \dots a_n$, then p divides atleast one of $a_1, \dots, a_n$ (Hint: Statement is true at $n = 2$).
5. $x^n - y^n$ is divisible by $x - y$ for all integers x, y.
6. If n is odd, then $x^n + y^n$ is divisible by $x + y$.
7. $4^{2n+1} + 3^{n+2}$ is divisible by 13.
8. $3^{3n+3} - 26n - 27$ is divisible by 169.
9. $5^{2n} - 6n + 8$ is divisible by 9.

*10. Prove that for $n \geq 3$, there are **odd** positive integers x and y such that $2^n = 7x^2 + y^2$.

2 Divisiblity and primes

1. Find all positive integers n such that $n + 2 \mid n^2 + 4$.
2. Prove that $4 \mid 5^n - 1$ for all n (You may use problem 5 in the above section).
3. Prove that $29 \mid 2^{466} + 5^{466}$ (Use problem 6 of above section).
4. Suppose that p and $p+2$ are both primes. Such pairs of primes are called **twin primes**. For example, the pairs $(3, 5)$, $(5, 7)$, $(11, 13)$, $(17, 19)$ are twin primes. Prove that if p and $p+2$ are twin primes with $p > 3$, then the number $p + 1$ is divisible by 6.

*5. Suppose that p, $p + 2$ and $p^2 + 2p - 2$ are all prime numbers. Prove that $p = 3$.

6. Let a and b be positive integers such that $a + b + ab = 2020$. Prove that $a + b = 88$.
7. Let a and b be positive integers such that $a + b = ab$. Prove that $a = b = 2$.

*8. Suppose that $2^n - 1$ is prime. Then n is prime.

*9. If $n^4 + 4^n$ is prime, prove that $n = 1$.

3 GCD, LCM and Euclid's Algorithm

1. Suppose that $c \mid a$ and $c \mid b$. Then $c \mid (a, b)$.
2. Suppose that $a \mid c$ and $b \mid c$. Then $[a, b] \mid c$.
3. Find all positive integers a and b such that $a + b + [a, b] = 25$.

*4. A point (x, y) in the XY plane is called an integer point if both x and y are integers. How many integer points lie on the line joining the points $(0, 0)$ and $(88, 64)$.

*5 Using Euclid's algorithm, prove that $(2^n - 1, 2^m - 1) = 2^{(n,m)} - 1$, where (n, m) denotes the GCD of m and n.

4. Find GCD of the following pairs of numbers using the Euclid's algorithm:

$$(68, 263),\ (72, 196),\ (1000, 5505),\ (36, 333)$$

Problems marked with * are difficult.