Common Measures of Magnitudes

Based on Euclid's Elements

Amritanshu Prasad

6 May 2014

Detail from a fresco by Rafael (circa 1510)

Euclid of Alexandria

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Flourished: 300BC

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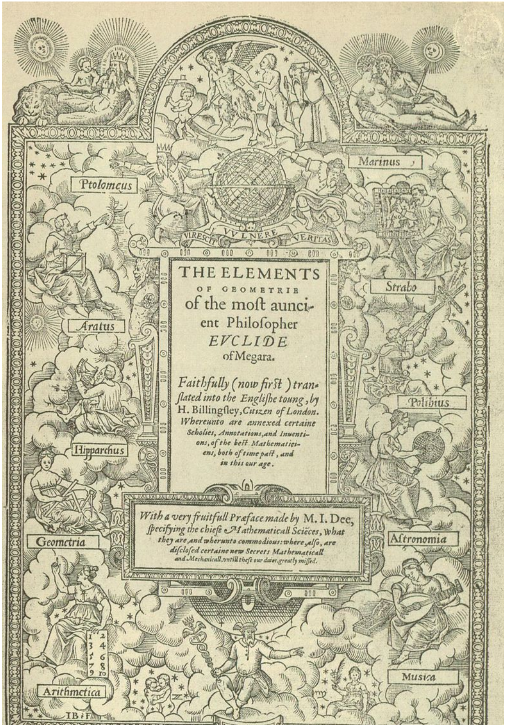
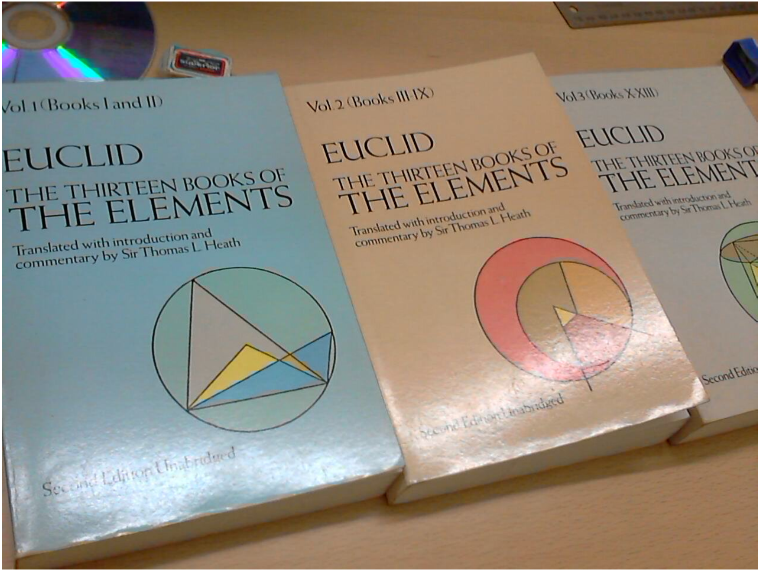
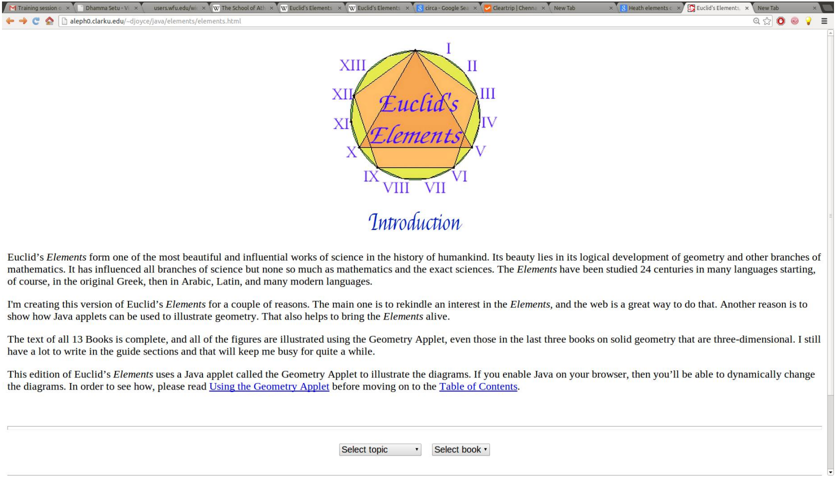


Cover page of the first English edition of Eulid's elements (Billingsley, 1570)

The version most commonly used these days is Heath's translation:



Or Joyce's online edition based on Heath:



Introduction

Euclid's *Elements* form one of the most beautiful and influential works of science in the history of humankind. Its beauty lies in its logical development of geometry and other branches of mathematics. It has influenced all branches of science but none so much as mathematics and the exact sciences. The *Elements* have been studied 24 centuries in many languages starting, of course, in the original Greek, then in Arabic, Latin, and many modern languages.

I'm creating this version of Euclid's *Elements* for a couple of reasons. The main one is to rekindle an interest in the *Elements*, and the web is a great way to do that. Another reason is to show how Java applets can be used to illustrate geometry. That also helps to bring the *Elements* alive.

The text of all 13 Books is complete, and all of the figures are illustrated using the Geometry Applet, even those in the last three books on solid geometry that are three-dimensional. I still have a lot to write in the guide sections and that will keep me busy for quite a while.

This edition of Euclid's *Elements* uses a Java applet called the Geometry Applet to illustrate the diagrams. If you enable Java on your browser, then you'll be able to dynamically change the diagrams. In order to see how, please read Using the Geometry Applet before moving on to the Table of Contents.

Select topic | Select book

From *A History of Mathematics* by Carl B. Boyer:

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However, the axioms used by mathematicians today are different, and are based on a careful analysis of language.

In this Lecture

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The arrangement of ideas is from the *Elements* but the emphasis is on discussing ideas, and not on giving Euclid-style proofs of theorems.

Commensurable Magnitudes

Definition (Elements, Book X, Definition 1)

Those magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure.

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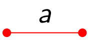
What does it mean to be “measured by a measure”?



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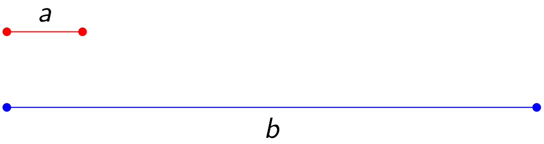

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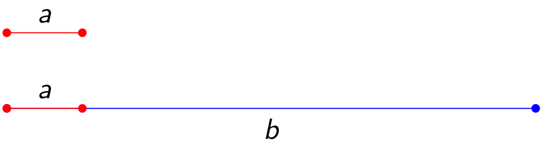

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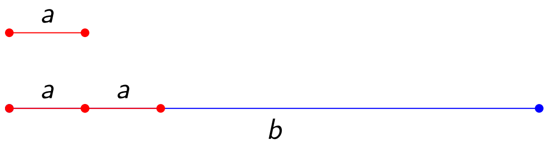

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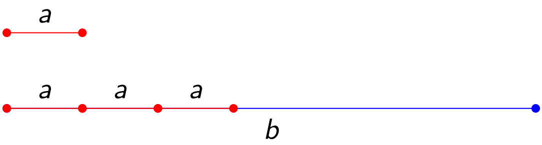

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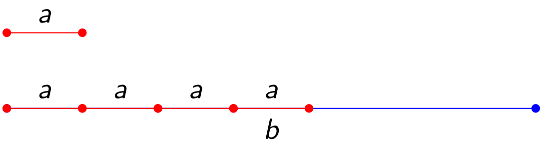

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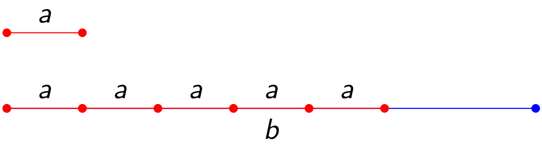

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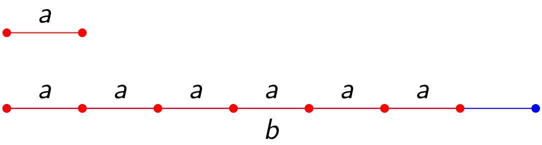

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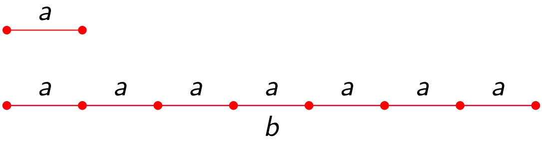

The magnitude a measures the magnitude b ($b = 7a$)

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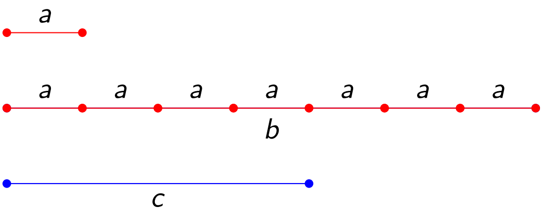

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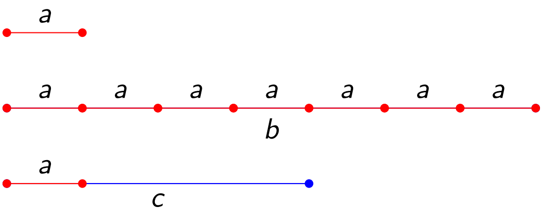

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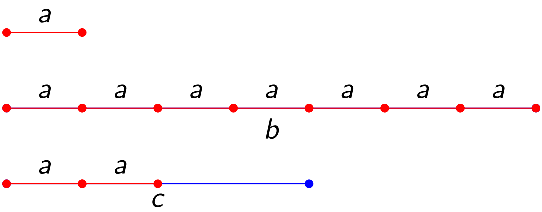

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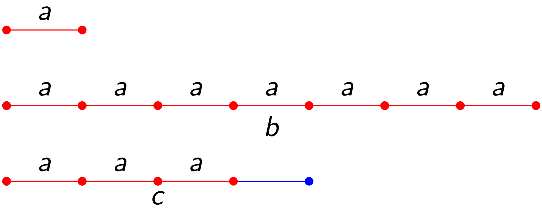

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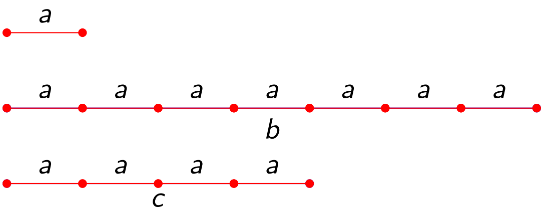

The measure a is a common measure of b and c.

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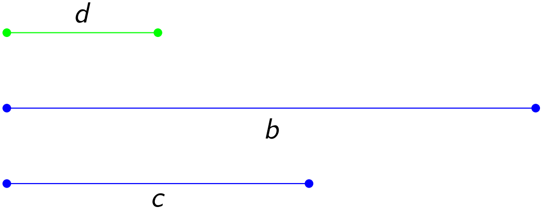

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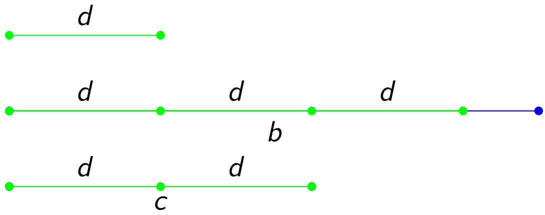

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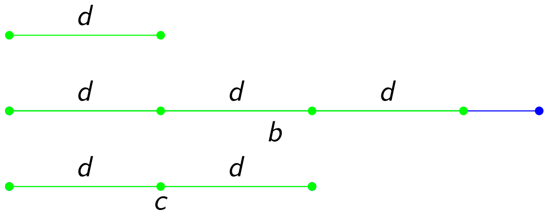

d measures b, but not c.

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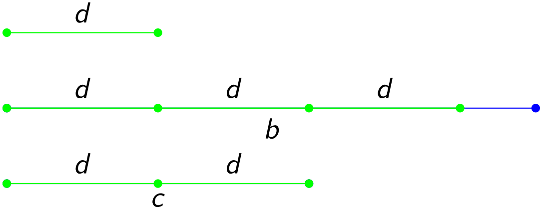

d measures b, but not c.
So d is not a common measure of b and c.

Commensurability and Rational Numbers

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If magnitudes a and b are commensurable then the ration a/b is a rational number.

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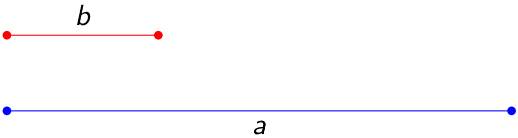
m and n are integers. □

What happens when b does not measure a?



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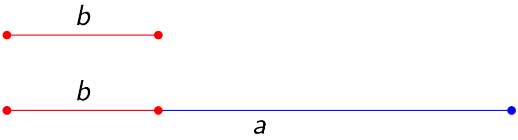

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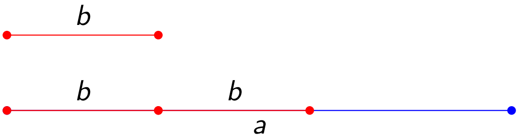

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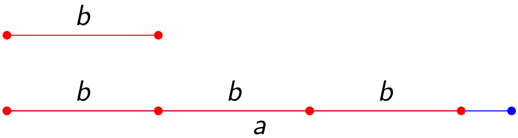

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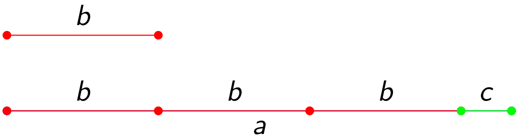

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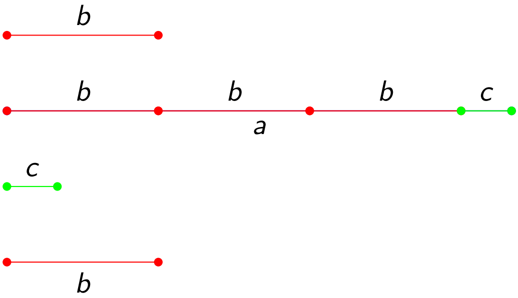
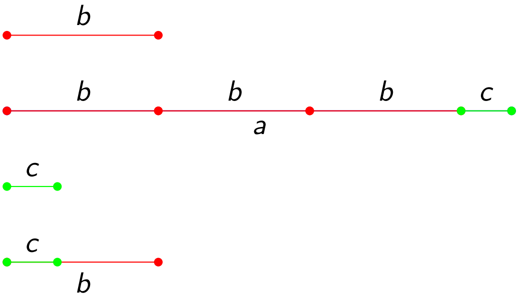


$$a = 3b + c$$

What happens when b does not measure a?



$$a = 3b + c$$

$$c < b$$

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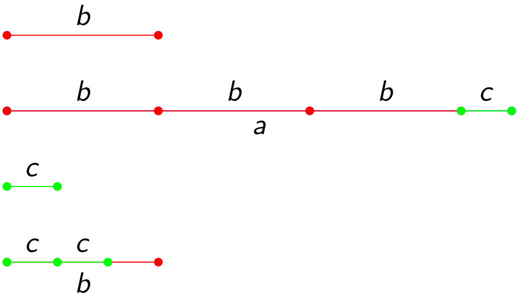

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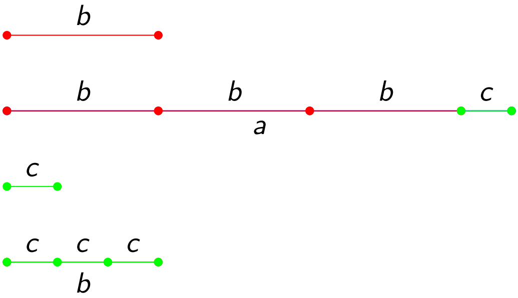

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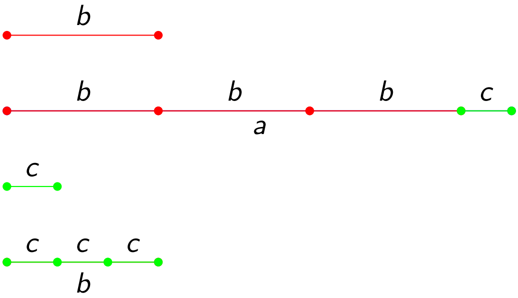

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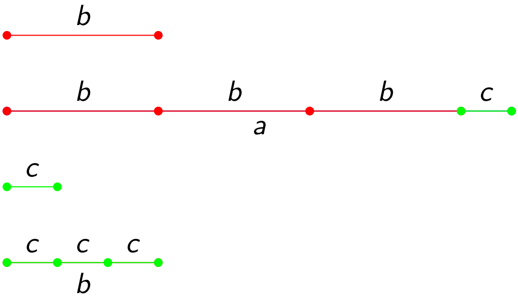
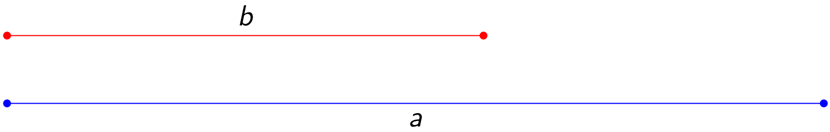


$$a = 3b + c$$

$$b = 3c$$

$$a = 3b + c = 3(3c) + c = 10c$$

Another Example



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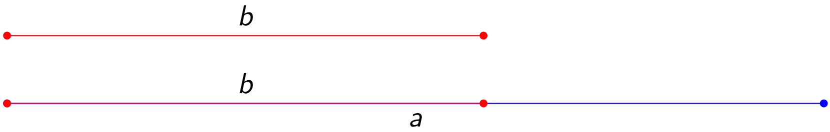

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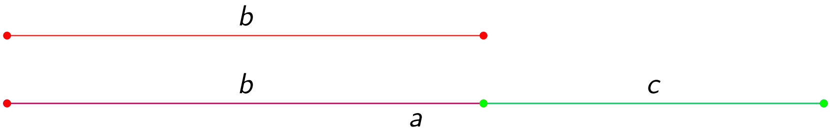

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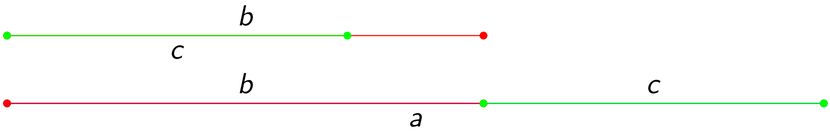

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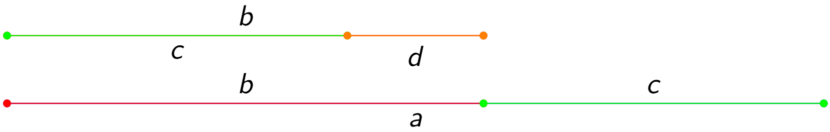

$$a = b + c$$

$$b = c + d$$

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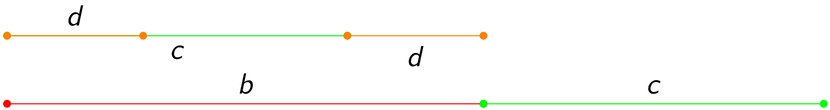

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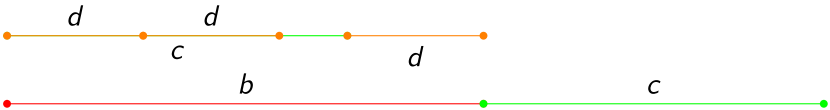

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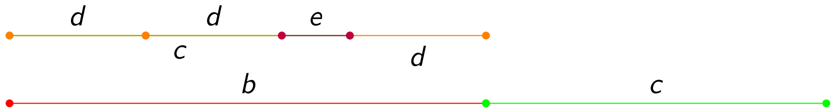
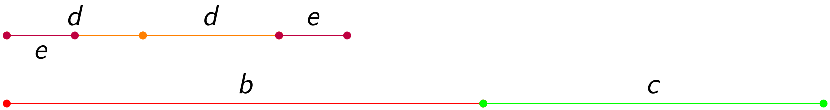


$$a = b + c$$

$$b = c + d$$

$$c = 2d + e$$

Another Example



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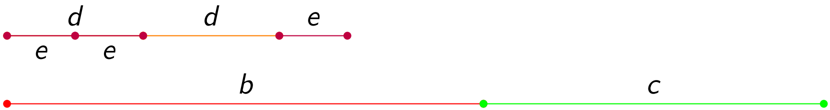

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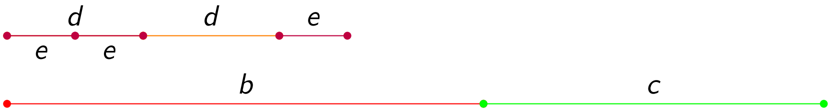
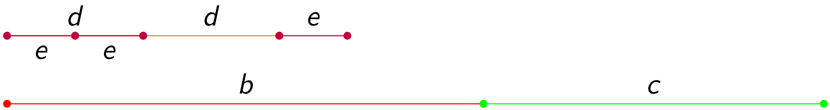


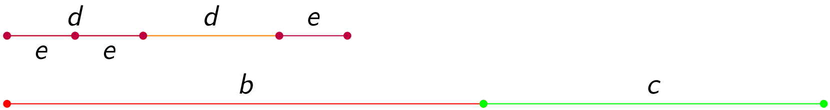
$$a = b + c$$
$$b = c + d$$
$$c = 2d + e = 5e$$
$$d = 2e$$

Another Example



$$a = b + c$$
$$b = c + d = 7e$$
$$c = 2d + e = 5e$$
$$d = 2e$$

Another Example



$$a = b + c = 12e$$
$$b = c + d = 7e$$
$$c = 2d + e = 5e$$
$$d = 2e$$

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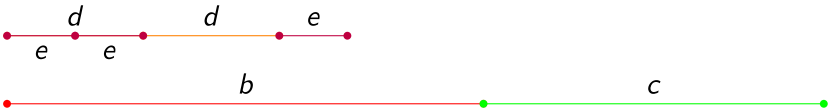

$$a = b + c = 12e$$
$$b = c + d = 7e$$
$$c = 2d + e = 5e$$
$$d = 2e$$

So e is a common measure of a and b, and furthermore, the ratio $a : b$ is $12 : 7$.

Consider the equations from the previous slide:

$$a = b + c \qquad c < b \tag{1}$$

$$b = c + d \qquad d < c \tag{2}$$

$$c = 2d + e \qquad e < d \tag{3}$$

$$d = 2e \tag{4}$$

Consider the equations from the previous slide:

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$$d = 2e \tag{4}$$

The equation (1) tells us:

$$\frac{a}{b} = 1 + \frac{c}{b}; \quad \frac{c}{b} < 1,$$

Consider the equations from the previous slide:

$$a = b + c \qquad c < b \tag{1}$$

$$b = c + d \qquad d < c \tag{2}$$

$$c = 2d + e \qquad e < d \tag{3}$$

$$d = 2e \tag{4}$$

The equation (1) tells us:

$$\frac{a}{b} = 1 + \frac{1}{b/c}; \quad \frac{b}{c} > 1.$$

Similarly, the equations (2), (3) and (4) tell us:

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Putting these together gives *the continued fraction expansion*:

$$\frac{12}{7} = \frac{a}{b} = 1 + \frac{1}{1 + \frac{1}{2+\frac{1}{2}}}.$$

Euclid's Theorem

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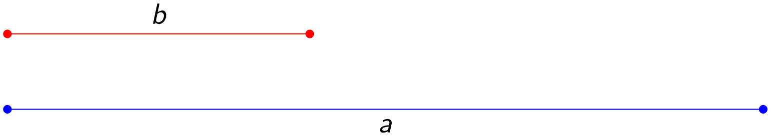
Euclid's algorithm, if it ends, gives a common measure for the magnitudes a and b.
Euclid proved that if this algorithm does not end, then a and b can not have a common measure, i.e., they are *incommensurable*.

Theorem (Book X, Proposition 2)

If, when the less of two unequal magnitudes is continually subtracted in turn from the greater that which is left never measures the one before it, then the two magnitudes are incommensurable.

The Key to Understanding Euclid's Theorem



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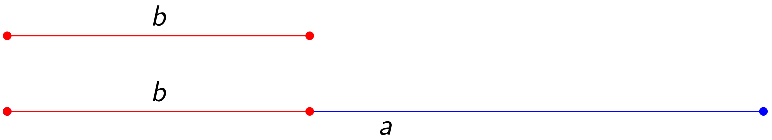

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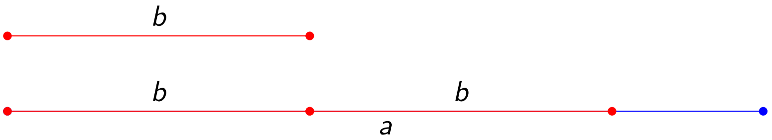

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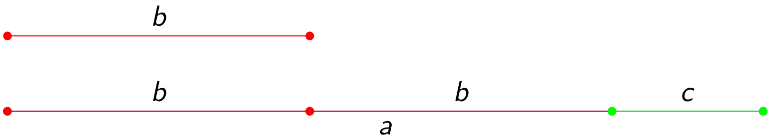

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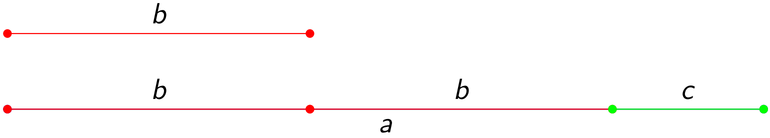

- $c < a/2$, so if Euclid's algorithm continues indefinitely, the magnitudes a, b, c, etc., diminish to become smaller than any given magnitude.

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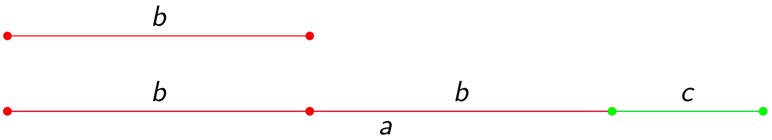

- $c < a/2$, so if Euclid's algorithm continues indefinitely, the magnitudes a, b, c, etc., diminish to become smaller than any given magnitude.
- The common measures of a and b are the same as the common measures of b and c.

The Proof of Euclid's Theorem

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Suppose that a and b have a common measure; call it x.
Then x is a common measure of all the magnitudes a, b, c, etc., obtained by Euclid's algorithm.
However, these will eventually diminish to become smaller than x, which is absurd.

Relation to Rational Numbers

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Theorem (Reinterpretation of Euclid's theorem)

Rational numbers are precisely those numbers which have finite continued fraction expansions.

Existence of Non-Commensurable Magnitudes

Theorem (Book X, Proposition 9)

The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares which have to one another the ratio which a square number has to a square number also have their sides commensurable in length. But the squares on straight lines incommensurable in length do not have to one another the ratio which a square number has to a square number; and squares which do not have to one another the ratio which a square number has to a square number also do not have their sides commensurable in length either.

Illustration of the Last Point

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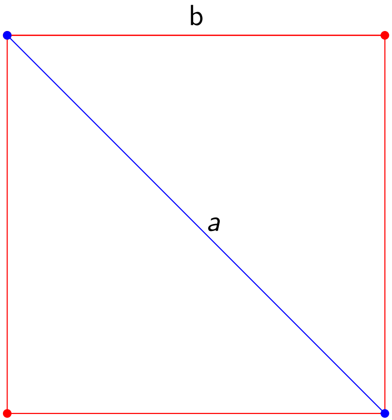

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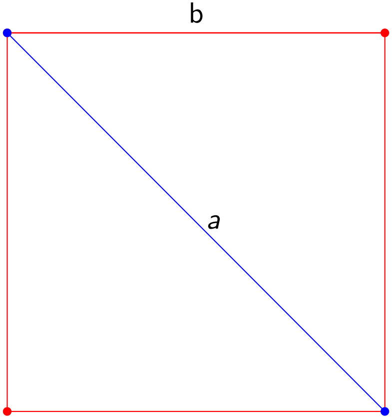

By Pythagoras's theorem:
$a^2 = 2b^2$.

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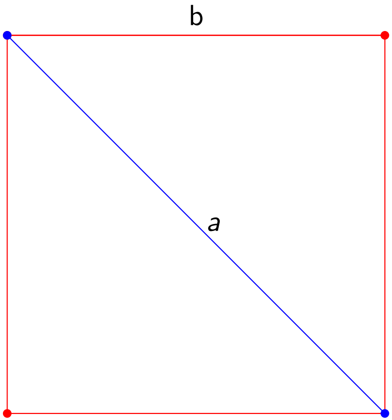

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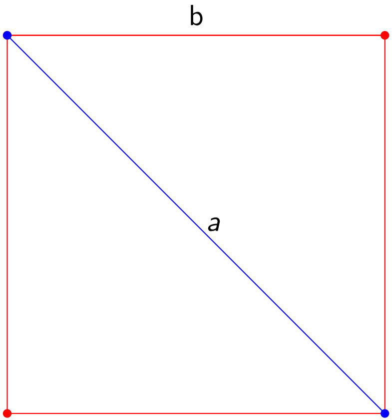

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$a^2 = 2b^2$.
So the squares of a and b do not have to one another a ratio which is a square number.
Euclid's theorem says that a and b are not commensurable, or in other words, $\sqrt{2}$ is an irrational number.

Continued Fraction Expansion of $\sqrt{2}$

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$$\sqrt{2} = 1 + (\sqrt{2} - 1)$$

Continued Fraction Expansion of $\sqrt{2}$

$$
\begin{aligned}
\sqrt{2} &= 1 + (\sqrt{2} - 1) \\
&= 1 + \frac{1}{\frac{1}{\sqrt{2}-1}}
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&= 1 + \frac{1}{2 + (\sqrt{2} - 1)} \\
&= 1 + \frac{1}{2 + \frac{1}{\frac{1}{\sqrt{2}-1}}} \\
&= 1 + \frac{1}{2 + \frac{1}{2+\frac{1}{\sqrt{2}-1}}} \text{ etc.}
\end{aligned}$$