Basic Combinatorics - Summer Workshop

2014

Chapter 1

Fundamental Principle of Counting 1

1.1 Introduction

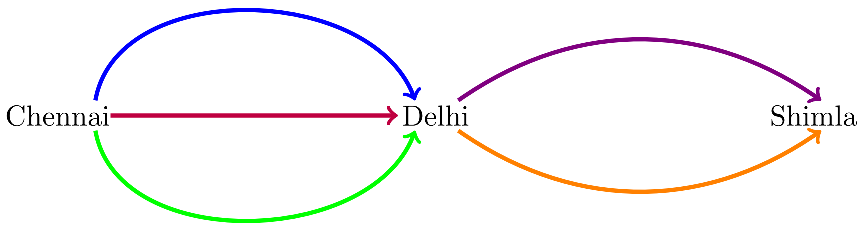
: We introduce this concept with a very simple example:

Example 1.1.1. I want to go by train from Chennai to Delhi and then from Delhi to Shimla.

- There are three trains say A, B and C from Chennai to Delhi.
- There are two trains, say R and S from Delhi to Shimla.

In how many ways can I complete this journey?

Proof. The figure below shows us the possible options.

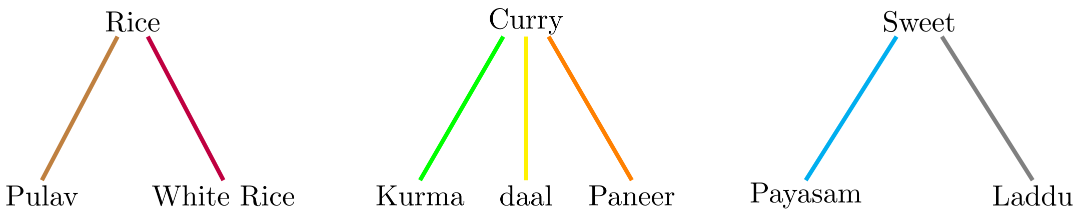


1. To go from Chennai to Delhi, we have to choose from three trains. So, there are 3 ways to go from Chennai to Delhi.
2. Suppose we took a train say A from Chennai to Delhi, we now have to take a train from Delhi to Shimla. So we have to choose from two trains R and S. We have 2 ways to go from Delhi to Shimla.

For each of the 3 trains we chose for going from Delhi to Chennai, we had a choice of two trains from Delhi to Shimla. So the total number of choices we have for the travel is $3 \times 2 = 6$. □

Example 1.1.2. Here is another example. I am having a meal at a hotel. I have to eat exactly three items: 1 rice item, 1 curry and 1 sweet. I have 2 choices of rice say Pulav and white rice, 3 choices for curry, say Kurma, Daal and paneer and 2 choices of sweets, say ladoo and payasam. How many choices of a meal do I have?

Proof. The following diagram shows us the choices we have.



My task is to select a meal.

- As I have 2 choices for rice, there are 2 ways to select the rice item.
- Suppose I took pulav, I have to choose a curry, there are 3 choices for selecting a curry.
- Now that I have selected my curry, I have to choose a sweet, there are 2 choices for selecting a sweet.

So For each choice of rice, we had a choice from 3 curry $\times$ 2 sweets. Hence, the total number of choices we have for the meal is: $2 \times 3 \times 2 = 12$. □

Example 1.1.3. Suppose I have a pack of playing cards. I have to just pick up a diamond or a heart. How many choices do I have?

Proof. There are 13 hearts and 13 diamonds. We have to choose exactly either diamond OR a heart. So we have 13 choices to choose a card from diamonds and 13 choices to choose from hearts.

- If we chose diamonds, we have 13 ways to pick the card.
- If we chose hearts, here too we have 13 ways to pick the card.

So we have a total of $13 + 13 = 26$ choices. □

Proposition 1.1.4. There are k tasks, T1, T2 and so on upto Tk. Suppose

- If T1 can be done in $m1$ ways.
- T2 can be done in $m2$ ways and so on,
- Tk can be done in mk ways,

1. If have to do all the tasks T1, T2, and so on upto Tk, then I have

$$m_1 \times m_2 \times \cdots \times m_k$$

ways to finish all these tasks one by one.

2. If I am supposed to do exactly one out of these k tasks, then I can do it in

$$m_1 + m_2 + \cdots + m_k$$

ways to do it.

This is called the **Fundamental Principle of Counting.**

As we saw, examples 1.1.1 and 1.1.2 explained how statement 1 of the above proposition works.
Example 1.1.3 explained how statement 2 of the above proposition works.

Exercises

Exercise 1.1.1. I am a captain of a cricket team. There is a match tomorrow for which one spin bowler and one fast bowler are needed.

1. There are 4 options of spin bowlers available to choose from.
2. There are 3 options of fast bowlers available to choose from.

How many choices of pairs of bowlers(spin bowler and fast bowler) do I have for selection?

Exercise 1.1.2. I am at a hotel. I want to have only one drink. I can either have a cool-drink or a milk-shake. There are 3 cool drinks available and 5 milk-shakes available. How many options are there for me?

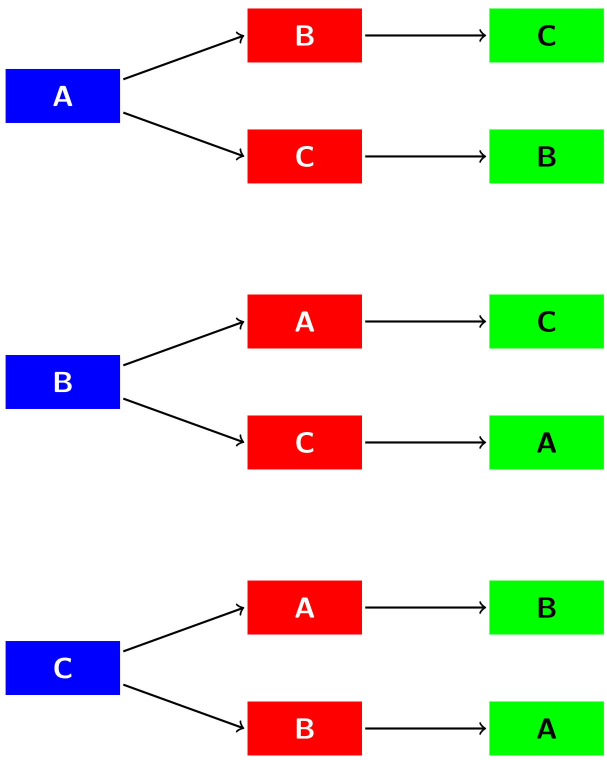
1.2 Permutations and Combinations

1.2.1 Arrangements

Like the last section, this section too will be introduced with a simple example.

Example 1.2.1. In a school, there are three children called A, B and C. There are 3 sports teams, Blue, Red and Green. We need to put one child in team Blue. From the remaining two, we have to put one child in team Red and the remaining child will go into team Green. In how many ways can these three children be put in these three teams?

Proof. The figure below shows the possible choices of selection.



- Task 1: Team Blue has to select a child. It has a choice of 3 children to select from.
- Once team Blue selects a child, there are two children left. So, we proceed to Task 2: Team Red has to select a child. It has a choice of two children to select from.
- Now it is time for the third task. Task 3: Team Green has to select a child. It has only one choice.

So by applying the fundamental principle of counting, which is mentioned in the previous section, we have a total of $3 \times 2 \times 1 = 6$ possible choices of selection. □

Definition 1.2.2. The above example motivates us to define this number. The number of ways, one can arrange n objects in n slots is

$$n \times (n-1) \times \cdots \times 2 \times 1$$

and this number is called n-**Factorial**. The symbol for this number is $n!$.

In the example 1.2.1 we are arranging 3 objects(children) in 3 slots(the 3 teams).

1.2.2 Combinations

In this section we talk about selecting objects from a bunch of objects. To see what this section is about, we will see an example.

Example 1.2.3. There are 5 cycles in a cycle shop. Say A, B, C, D, E. I want to buy 3 cycles. How many options do I have?

Solution. There are 5 options for selecting the first cycle.
4 options for selecting the second cycle and
3 options for selecting the third cycle.
Which means that the number of choices is $5 \times 4 \times 3 = 60$. But does this 60 count any selection more than once? It does. Why?
Suppose I chose the cycles B, D and E. Then these are the ways in which i could have selected the cycles B, D and E:

- B-D-E
- B-E-D
- D-B-E
- D-E-B
- E-B-D
- E-D-B

When B, D and E were selected: We had a choice of 3 to make the first selection. Once the first one among B, D and E is chosen, we are left with a choice of 2 cycles for selecting the second cycle and one choice for selecting the third. Hence, the choice B, D and E is repeated $3 \times 2 \times 1 = 6$ times. This happens for each choice.
So, the number of choices mentioned earlier(which is 60) has to be divided by 6. So the total number of choices for selection is

$$\frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$$

This example leads to the following definition:

Definition 1.2.4. If there are n different objects and we want to pick up only r of them($r < n$), then the number of options of selection is

$$\frac{n \times (n-1) \times \cdots \times (n-r+1)}{r \times (r-1) \times \cdots \times 2 \times 1}$$

and this number is called n "choose" r. It is given the symbol $\binom{n}{r}$

Now we shall look at selections where the order of selections matters.

Example 1.2.5. There are five biscuits A, B, C, D and E and three boxes labelled "1", "2" and "3". I choose a biscuit and put it in box "1", then select the next biscuit from the remaining 4 and put it in box "2" and select the 3rd biscuit from the last 3 and put it in box "3". How many such arrangements are possible?

Solution.

- Like in the last problem, to choose the first biscuit, we have 5 biscuits to choose from. But now, we put that into box "1".
- There are 4 biscuits left, so we have 4 of them to choose from and put in box "2".
- Finally, there are only 3 biscuits left. So we have to choose one from those 3 and put in box "3".

So, by the fundamental principle of counting, there are $5 \times 4 \times 3 = 60$ such arrangements possible. This leads us to define the following:

Definition 1.2.6. The number of ways to arrange r objects out of a collection of n different objects is equal to

$$n \times (n-1) \times \cdots \times (n-r+1)$$

This number is called n"permutation"r and the symbol for this number is ${}^nP_r$.
The word *permutation* is another meaning of the word arrangement.

Exercises

Exercise 1.2.1. There are 4 boxes with numbers "1" , "2", "3" and "4" written on them. There are 4 balls of different colours, blue, green, yellow and red. The task is to put one ball cup 1, another in cup 2, another in cup 3 and the remaining one ball in cup 4. In how many ways can those 4 balls be put in those 4 cups?

Exercise 1.2.2. How many four-lettered words can be formed with the letters D O S A?

Exercise 1.2.3. Five people A, B, C, D and E have been allotted seats 1, 2, 3, 4, 5 in a train. How many possible seating arrangements are there?

Exercise 1.2.4. There is a basket with four fruits, Apple, Mango, Banana and Orange. I want to pick up two fruits, one after the other. How many combinations of fruits can I choose from?

Chapter 2

Fundamental Principle of Counting 2

2.1 Arrangements of Repeated Objects

Here we look at some tricky but fun problems. In the last chapter, we looked at arrangments of a bunch of different objects. But what if we approach a problem in which we have to look at bunch of objects where some repeat, for example: a bag of 10 fruits which contains 2 apples, 4 mangoes, 3 bananas and 1 mosambi. etc.
To understand things better, let us look at this example:

Example 2.1.1. Consider the word S U C C E S S. I want to form 7-lettered words with the letters S, U, C, C, E, S and S. How many words can I form?

Solution. Given that the word SUCCESS has 7 letters, we might immediately say that we can form $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7!$ words with its letters. But, are we counting any word more than once? yes, we are. To see why, let us put the 2 E's and the 3 S's in different colours say C and C and S, S and S.
Now we have the word SERIES itself written as:

S U C C E S S   S U C C E S S

S U C C E S S   S U C C E S S

S U C C E S S   S U C C E S S

S U C C E S S   S U C C E S S

S U C C E S S   S U C C E S S

S U C C E S S   S U C C E S S

As you can see above that the three S's can be put in their 3 places in $3 \times 2 \times 1 = 6 = 3!$ ways and the two C's can be put in their two places in $2 \times 1 = 2!$ ways. So, these two tasks together can be done in $6 \times 2 = 12$ ways as you can see above. So we have this word SUCCESS repeating $3! \times 2!$ times. So, we have to divide the number $7!$ by $3! \times 2!$. The number of words we can actually form is

$$\frac{7!}{3! \times 2!}$$

Example 2.1.2. Here is another example. There is a chess tournament. There are 3 Indian players, 2 German players and 2 players from Iran. The tournament got over and the results list only showed the name of each player's country with his/her in order of the rank. For example

Rank	Country
1	India
2	Iran
3	India
4	Germany
5	India
6	Iran
7	Germany

Then how many such results are possible?

Solution. There are 7 players, so we may again say that there are 7! results possible. But we will again ask the question: Are some results being counted more than once? The answer is Yes! But why?
Now, the results table only shows the name of the country of the player with his/her rank. In any chart, India appears at three ranks, Iran and Germany appear in two ranks each.

So, those three ranks of the Indian players could be filled in 3! ways.

The two ranks of the Iran players could be filled in 2! ways and

the two ranks of the German players can be filled in 2! ways.

Hence, each rank table repeats $3! \times 2! \times 2!$ times. So, the possible number of results is

$$\frac{7!}{3! \times 2! \times 2!}.$$

Now we shall look at a general formula for such problems.

Definition 2.1.3. If we have n objects where m_1 are alike, m_2 are alike, ... and m_r are alike. Then the number of ways to arrange these objects in a line is

$$\frac{n!}{m_1! \times m_2! \times \cdots \times m_r!}$$

Now, there are arrangements of n objects with m_1 being alike, m_2 being alike,... and m_k being alike. So there is this question of how many ways are there to arrange these objects so that ojects of the same kind are always together.
To understand this question, let us look at an example:

Example 2.1.4. There are 2 boys and 2 girls. A teacher wants them to sit on a bench in such a way that the boys are always sitting beside each other and the girls are always sitting beside each other. In how many ways can we make the seating arrarngments.

Solution. We could have the girls sitting on the left and the boys to their right OR else we could have the boys sitting on the left and the girls sitting on the right. So there are $2! = 2$ ways to decided who occupies the first 2 places and who occupies the next 2 places. Let the girls be called G1 and G2. Let the boys be called B1 and B2. Consider any one ordering of the girls and boys: Say girls on the left and boys on the right. Then these are the possible seating arrangements we have:

G1 G2 B1 B2 G1 G2 B2 B1

G2 G1 B1 B2 G2 G1 B2 B1

So, the girls can be arranged in their places in $2! = 2$ ways and the boys also are arranged in their places in 2! ways. So we have $2! \times 2!$ arrangements for this ordering of girls and boys.
So totally, we have $2! \times (2! \times 2!) = 8$ seating arrangements.

Exercises

Exercise 2.1.1. Find the number of words that can be formed using the letters of the word A D Y A R.

Exercise 2.1.2. I have 10 Uno cards with me of which there are 4 blue cards, 3 red cards, 2 yellow cards and one green card. I just want to arrange all these cards in a line from left to right. In how many ways can I do so?

Exercise 2.1.3. (Fun Exercise) Find the number of words that can be formed using the letters of the word M I S S I S S I P P I.

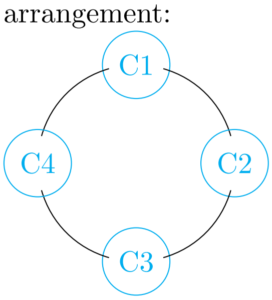
Exercise 2.1.4. In a TV show-room, there are 4 Samsung TV's, 2 Lg TV's and 3 Sony TV's. The shopkeeper wants to arrange them in such a way that all the Samsung TV's are beside each other, all the Lg TV's are beside each other and all the Sony TV's are beside each other. How many ways can he arrange these TV's?

2.2 Circular Permutations

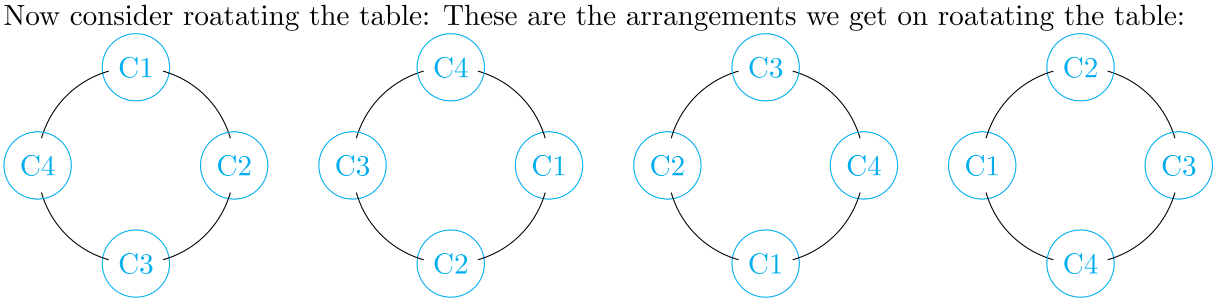
In the previous sections, we have seen how to arrange objects in a straight line. But there are problems of counting arrangements of objects around a circle. Let us look at one such example:

Example 2.2.1. There are 4 people C1,C2, C3 and C4. We want to make them sit around a round table. How many such seating arrangements are possible?

Solution. There are 4 places around that table that need to be filled. We might easily say that the number of arrangements is $4! = 24$. But then, are we counting a seating arrangement more than once? The answer is yes. Now you may ask why? Consider the following seating arrangement:



Now consider roatating the table: These are the arrangements we get on roatating the table:



Observe in the diagrams above:

- C2 is always sitting to the left of C1,
- C3 is always sitting to the left of C2,
- C4 is always sitting to the left of C3 and,

- C1 is always sitting to the left of C4.

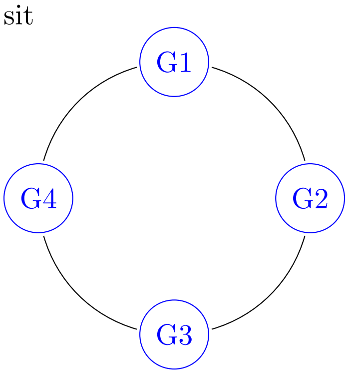
So these arrangements are the same. Which means that this arrangement has been counted 4 times. So we have $\frac{4!}{4} = 3!$ arrangements.

Proposition 2.2.2. In general, if we have to arrange n distinct objects around a circle, we have $(n-1)!$ possible arrangements.

This is one kind of a counting problem. There is another in which we have two or more types of objects with certain restrictions on arrangement. Like how to make m-boys and n-girls sit around a round table in such a way that no two boys are sitting together. Here is an example:

Example 2.2.3. There are four girls G1, G2, G3 and G4 and three boys B1, B2 and B3. We want to make them sit around a round table in such a way that any no two boys sit beside each other. How many possible arrangements are there of this kind.

Solution. First we make the girls sit. From the way we did the previous example, there are $(4-1)! = 3!$ ways to make them sit. The diagram below shows one such way to make the girls sit



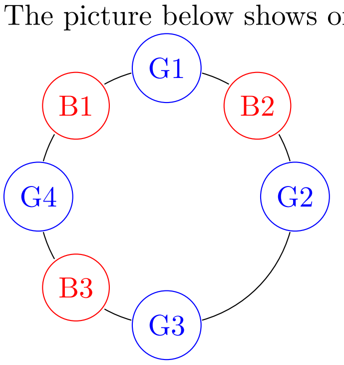
Now, we have to make the boys sit. In the question, we didn't want any two boys to sit together. This means that each of the boys has to sit in between two girls. So, from the picture above, each boy has to sit in one of those empty seats in between the girls.
We see that

Boy B1 has a choice of 4 seats to choose from.

Once B1 has taken his seat, there are 3 seats left, so B2 has 3 seats to choose from.

Once B2 is seated, there are two seats left. So B2 has a choice of 2 seats to choose from.

The picture below shows one such arrangement of the boys and the girls:



So we have a total of $4 \times 3 \times 2 = 24$ ways to make the boys sit.
This is for a given seating arrangement of the girls. So there are $3! \times (4 \times 3 \times 2)$ seating arrangements available.

Exercises

Exercise 2.2.1. Five men want to play a game of cards around a table. In how many ways can we make these men sit around the table?

Exercise 2.2.2. 4 ladies and their respective husbands are at a restaurant. They want to sit around a dining table in such a way that each husband is next to his wife. How many such seating arrangements are available?