Puzzles

By

ANIL KUMAR C P

The Institute of Mathematical Sciences, Chennai

Puzzles for

kids

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To my School Teachers

Gurur Brahma Gurur Vishnu, Gurur Devoh Maheswaraha
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Contents

An equation means nothing to me unless it expresses a thought of God

- Srinivasa Ramanujan

Chapter 1

Magic Squares

A magic square is an arrangement of objects which can be added in the form of a square grid such that the sum of the objects in any row, in any column, the diagonal and the anti diagonal match.

Here are some examples.

Example 1.1. 3×3 grid of numbers

8	1	6
3	5	7
4	9	2

The sum is always 15.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

The sum is always 65.

Here is a 3×3 grid of ordered pairs.

(8,12)	(1,19)	(6,14)
(3,17)	(5,15)	(7,13)
(4,16)	(9,11)	(2,18)

. The sum is always $(15, 45)$.

Here is a grid of variables which can be added.

A+P	B+Q	C+R	D+S
C+S	D+R	A+Q	B+P
D+Q	C+P	B+S	A+R
B+R	A+S	D+P	C+Q

The sum is always $A + B + C + D + P + Q + R + S$.

Chapter 2

Operations on Magic Squares

Operations that can be performed on magic squares.

Multiplication by a constant, adding / subtracting a constant, Reflection about the central horizontal line, central vertical line, diagonal, anti diagonal. Addition of two magic squares.

Usage of Arithmetic progressions. There exists arbitrary large odd size magic squares containing only prime numbers.

Chapter 3

Construction of Odd Size Magic Squares

3.0.1 Construction of $(2n+1) \times (2n+1)$ Magic Square

Construct a $(2n+1) \times (2n+1)$ magic square of ordered pairs as follows: Write (i, i) in the middle column for $i = 0, \ldots, 2n$ from top to bottom. Add $(1, 2) mod\ (2n+1)$ and wrap around in each row. We get a magic square of ordered pairs which in base n notation is the usual $(2n+1) \times (2n+1)$ magic square containing numbers $0, \ldots, (n^2 - 1)$. The diagonal and the antidiagonal sum is the same requires proof.

Example 3.1. For $n = 7$ we get the following.

(4,1)	(5,3)	(6,5)	(0,0)	(1,2)	(2,4)	(3,6)
(5,2)	(6,4)	(0,6)	(1,1)	(2,3)	(3,5)	(4,0)
(6,3)	(0,5)	(1,0)	(2,2)	(3,4)	(4,6)	(5,1)
(0,4)	(1,6)	(2,1)	(3,3)	(4,5)	(5,0)	(6,2)
(1,5)	(2,0)	(3,2)	(4,4)	(5,6)	(6,1)	(0,3)
(2,6)	(3,1)	(4,3)	(5,5)	(6,0)	(0,2)	(1,4)
(3,0)	(4,2)	(5,4)	(6,6)	(0,1)	(1,3)	(2,5)

$\xrightarrow{\text{Multiply by } (7,1)}$

(28,1)	(35,3)	(42,5)	(0,0)	(7,2)	(14,4)	(21,6)
(35,2)	(42,4)	(0,6)	(7,1)	(14,3)	(21,5)	(28,0)
(42,3)	(0,5)	(7,0)	(14,2)	(21,4)	(28,6)	(35,1)
(0,4)	(7,6)	(14,1)	(21,3)	(28,5)	(35,0)	(42,2)
(7,5)	(14,0)	(21,2)	(28,4)	(35,6)	(42,1)	(0,3)
(14,6)	(21,1)	(28,3)	(35,5)	(42,0)	(0,2)	(7,4)
(21,0)	(28,2)	(35,4)	(42,6)	(0,1)	(7,3)	(14,5)

$\xrightarrow{\text{Add both components}}$

29	38	47	0	9	18	27
37	46	6	8	17	26	28
45	5	7	16	25	34	36
4	13	15	24	33	35	44
12	14	23	32	41	43	3
20	22	31	40	42	2	11
21	30	39	48	1	10	19

$\xrightarrow{\text{Add 1}}$

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

We get the 7×7 magic square.

3.0.2 Construction of the 5×5 from the Grid Box Co-ordinates

This is one way to construct the magic square for an odd integer $n = 5$.

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)

$\xrightarrow{\text{Subtract } (0,1)}$

(1,0)	(1,1)	(1,2)	(1,3)	(1,4)
(2,0)	(2,1)	(2,2)	(2,3)	(2,4)
(3,0)	(3,1)	(3,2)	(3,3)	(3,4)
(4,0)	(4,1)	(4,2)	(4,3)	(4,4)
(5,0)	(5,1)	(5,2)	(5,3)	(5,4)

$\xrightarrow{\text{Keep a copy and Multiply a copy by } (1,2) \text{ component wise}}$

(1,0),(1,0)	(1,1), (1,2)	(1,2),(1,4)	(1,3),(1,6)	(1,4),(1,8)
(2,0),(2,0)	(2,1), (2,2)	(2,2),(2,4)	(2,3),(2,6)	(2,4),(2,8)
(3,0),(3,0)	(3,1),(3,2)	(3,2),(3,4)	(3,3),(3,6)	(3,4),(3,8)
(4,0),(4,0)	(4,1),(4,2)	(4,2),(4,4)	(4,3),(4,6)	(4,4),(4,8)
(5,0),(5,0)	(5,1),(5,2)	(5,2),(5,4)	(5,3),(5,6)	(5,4),(5,8)

$\xrightarrow{\text{Add } (\frac{n-1}{2} = 2, 0) \text{ to the first copy component wise and keep the second copy as it is}}$

(3,0),(1,0)	(3,1),(1,2)	(3,2),(1,4)	(3,3),(1,6)	(3,4),(1,8)
(4,0),(2,0)	(4,1),(2,2)	(4,2),(2,4)	(4,3),(2,6)	(4,4),(2,8)
(5,0),(3,0)	(5,1),(3,2)	(5,2),(3,4)	(5,3),(3,6)	(5,4),(3,8)
(6,0),(4,0)	(6,1),(4,2)	(6,2),(4,4)	(6,3),(4,6)	(6,4),(4,8)
(7,0),(5,0)	(7,1),(5,2)	(7,2),(5,4)	(7,3),(5,6)	(7,4),(5,8)

$\xrightarrow{\text{Sum the components in each copy}}$

3,1	4,3	5,5	6,7	7,9
4,2	5,4	6,6	7,8	8,10
5,3	6,5	7,7	8,9	9,11
6,4	7,6	8,8	9,10	10,12
7,5	8,7	9,9	10,11	11,13

$\xrightarrow{\text{Remainder when / by 5}}$

3,1	4,3	0,0	1,2	2,4
4,2	0,4	1,1	2,3	3,0
0,3	1,0	2,2	3,4	4,1
1,4	2,1	3,3	4,0	0,2
2,0	3,2	4,4	0,1	1,3

$\xrightarrow{\text{Multiply the first one by 5}}$

15,1	20,3	0,0	5,2	10,4
20,2	0,4	5,1	10,3	15,0
0,3	5,0	10,2	15,4	20,1
5,4	10,1	15,3	20,0	0,2
10,0	15,2	20,4	0,1	5,3

$\xrightarrow{\text{Add the II one to the I one}}$

16	23	0	7	14
22	4	6	13	15
3	5	12	19	21
9	11	18	20	2
10	17	24	1	8

$\xrightarrow{\text{Add 1}}$

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

We get the 5×5 magic square of numbers.

Chapter 4

Other Magic Squares

4.1 Birthday Magic Squares

Consider the magic square

A+P	B+Q	C+R	D+S
C+S	D+R	A+Q	B+P
D+Q	C+P	B+S	A+R
B+R	A+S	D+P	C+Q

Suppose a person A's birthday falls on 30/10/2005.

Then solve for A, B, C, D, P, Q, R, S using equations $A + P = 30$, $B + Q = 10$, $C + R = 20$, $D + S = 05$. We can choose $A = 22, B = 7, C = 15, D = 3, P = 8, Q = 3, R = 5, S = 2$ Then the magic square becomes

30	10	20	05
17	8	25	15
6	23	9	27
12	24	11	18

with first row as person A's birthday.

4.2 Multiplicative and Function Sum Magic Squares

Example 4.1. 3×3 grid of numbers

256	2	64
8	32	128
16	512	4

The product is always 32768.

64	1	36
9	25	49
16	81	4

What is the function?

630	280	1260
840	504	360
315	2520	420

What is the function?

Chapter 5

Representable and Non-representable Numbers of the Form $ax + by$

We say a number n is representable by positive integers a, b if there exists non-negative integers x, y such that $ax + by = n$.

Question 1. *Find a general formula for the highest number not representable by a, b. Find a formula for the number of non-representable numbers.*

Example 5.1. Let $a = 5, b = 7$. In the table below, in each column arithmetic progression the first multiple of 7 is marked.

$\overbrace{\underline{0}}$	1	2	3	4
5	6	$\overbrace{\underline{7}}$	8	9
10	11	12	13	$\overbrace{\underline{14}}$
15	16	17	18	19
20	$\overbrace{\underline{21}}$	22	23	24
25	26	27	$\overbrace{\underline{28}}$	29
30	31	32	33	34

The highest number not representable by $(5, 7)$ is 23. The total number of numbers from $0, 1, \ldots, 23$ that are not representable is 12 and the number of numbers which are representable is also 12.

Example 5.2. Let $a = 5, b = 9$. In the table below, in each column arithmetic progression the first multiple of 9 is marked.

$\overbrace{\underline{0}}$	1	2	3	4
5	6	7	8	$\overbrace{\underline{9}}$
10	11	12	13	14
15	16	17	$\overbrace{\underline{18}}$	19
20	21	22	23	24
25	26	$\overbrace{\underline{27}}$	28	29
30	31	32	33	34
35	$\overbrace{\underline{36}}$	37	38	39

The highest number not representable by $(5, 9)$ is 31. The total number of numbers from $0, 1, \ldots, 31$ that are not representable is 16 and the number of numbers which are representable is also 16.