TIME POINTS AND INTERVALS

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1. Time

Here are some "bookish" examples of temporal reasoning:

Galileo died in 1642 implies that Galileo was alive on 31 December 1641 and that Galileo was not alive on 1 January 1643.

A projectile fired into the sky implies it moves upwards for a while and then it moves downwards for a while (the predicate $PositiveHeight(t)$ is true for some times t and false for some other t).

Facts are timelessly true, the state of the world changes. Generations of schoolchildren have struggled with this kinematic view where time is reduced to a quantity, modelled in an appropriate number system. Typically the real numbers are chosen, because if you ask how a set of durationless time points can make up a duration, you can be told that uncountability makes it so, as if that explains everything.[1] There are three philosophical assumptions built into this "classical" model:

Time exists independent of change: Event e_1 precedes event e_2 is analysed as there exist times t_1, t_2 such that e_1 occurs at t_1, e_2 occurs at t_2 and t_1 precedes t_2. But it seems to make sense (and ties in with our intuitive experience) to have events with relations such as causation or succession among them (maybe equivalenced by simultaneity) and saying that t_1 precedes t_2 if each event occurring at time t_1 (eg, e_1 is a member of the simultaneity class t_1) precedes each event e_2 at t_2.

Time consists of uncountably many durationless points: This is the "time as river flow" metaphor, where the river is also seen as an aggregate of water drops (time points). Natural language shares this ambiguity: in "We spend a lot of time together" time is used as a mass noun, in "We see each other several times" it is used as a count noun.[2]

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[1] There are infinitely many order-preserving automorphisms on the real line which do not preserve length.

[2] Natural language has a much richer structure than can be expressed in our syntax. Try the following sentence, which doesn't seem to be expressible in a pure point or pure interval formalism at all: "Whenever he comes to Calcutta, he starts lecturing to us."

This point-based view agrees with the older view (cf, Aristotle) that time can be limitlessly divided[3], but reifies each potential point of division as a constituent of time. The older view was interval-based, and most of our historical temporal notions (days, weeks, months …) do correspond to intervals. A society without clocks would have nothing corresponding to 11:42 am. Is 11:42 a point or is it an atomic minute-long interval?

States are primitive, events derived: Each time is assigned a "state of the world". Change is a consequence of having different assignments at different times (cf, Russell).

How does one define the state of being married? Intuitively, it is a result of the event of getting married, which seems more primitive.

2. Models and logics

Let $Prop$ be a set of propositions. A point frame $\mathcal{T} = (T, <)$ is a nonempty strict linear order. A point model $M = (T, <, TV)$ is a point frame with a valuation $TV : T \to \wp(Prop)$ which says which propositions hold at a time point. The formulas of point tense logic are defined by closure under boolean operations and the future (F) and past (P) modalities.

Point formulas $\alpha ::= p \in Prop \mid \neg\alpha \mid \alpha_1 \vee \alpha_2 \mid \mathsf{P}\alpha \mid \mathsf{F}\alpha$

Satisfaction is inductively defined as usual. For example, $M, t \models \mathsf{P}\alpha$ if there is some $s < t$ such that $M, s \models \alpha$. Complete finite axiomatizations and decidability of validity are known (recall that the frames are strict linear orders).

An interval frame $\mathcal{I} = (I, \sqsubseteq, \ll)$ is a nonempty poset $(I, \sqsubseteq)$ with greatest lower bounds, together with a strict partial order $\ll$ which is *monotonic*, that is, if $w \sqsubseteq x \ll y$ then $w \ll y$, and if $x \ll y$ and $z \sqsubseteq y$ then $x \ll z$. The "glb" condition is formally stated using an *overlap* relation: let xOy if for some lb, $lb \sqsubseteq x$ and $lb \sqsubseteq y$. The condition is that if x and y have a lower bound then they have a greatest lower bound. That is, if xOy then for some glb, $glb \sqsubseteq x$, $glb \sqsubseteq y$ and for all lb such that $lb \sqsubseteq x$ and $lb \sqsubseteq y$, $lb \sqsubseteq glb$. We write this element as $glb(x, y)$.

An interval model $(I, \sqsubseteq, \ll, IV)$ assigns propositions to intervals, $IV : I \to \wp(Prop)$. The logical language above can be extended to interval tense logic which has the future and past modalities for the $\ll$ relation, and down ($\Diamond_{down}$) and up ($\Diamond^{up}$) modalities for the $\sqsubseteq$ relation.

Interval formulas $\varphi ::= p \in Prop \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \mathsf{P}\varphi \mid \mathsf{F}\varphi \mid \Diamond_{down}\varphi \mid \Diamond^{up}\varphi$

$M, x \models \Diamond_{down}\varphi$ if there is some $w \sqsubseteq x$ such that $M, w \models \varphi$.
$M, x \models \Diamond^{up}\varphi$ if there is some $y \sqsupseteq x$ such that $M, y \models \varphi$.

The monotonicity condition on interval frames can be defined by the axioms $\mathsf{F}\Diamond_{down}\varphi \supset \mathsf{F}\varphi$ and $\mathsf{P}\Diamond_{down}\varphi \supset \mathsf{P}\varphi$. The glb condition is not definable by these modalities. Halpern and Shoham (1991) showed that validity of interval tense logic is not decidable—not even recursively axiomatizable.

[3] We can get density by having no lower bound on the finite duration of an instant, but the model is bizarre.

What is the reason for this disparity? Following van Benthem (1983), we will examine the question from the point of view of models.

2.1. **Point frames to interval frames.** Given a point frame $\mathcal{T} = (T, <)$, we can construct an interval frame $Int(\mathcal{T}) = (I, \sqsubseteq, \ll)$ by letting I be the nonempty convex subsets of T, $\sqsubseteq$ be inclusion and $x_1 \ll x_2$ iff for every t_1 in x_1, t_2 in x_2, $t_1 < t_2$. This interval frame satisfies several additional properties, of which we mention atomicity: for every x there is $x_1 \sqsubseteq x$, such that if $x_0 \sqsubseteq x_1$ then $x_0 = x_1$.

Given an atomic interval frame $\mathcal{I} = (I, \sqsubseteq, \ll)$, we can construct a point frame $Pt(\mathcal{I}) = (T, <)$ by letting T be the set of atoms of I (like C_1 above) and $<$ be $\ll$ restricted to T.

Proposition 1 ([Ben1]). *$Int(Pt(\mathcal{I})) \cong \mathcal{I}$.*

Let us consider a more specific example. Since $\mathcal{T}$ is a strict linear order, we can define its intervals to be the usual "open intervals" $(t_1, t_2) = \{t \mid t_1 < t < t_2\}$. (We won't bother about the limit points.) We can now define $\sqsubseteq$ to be a "during" relation (t, u) *during* (v, w) if $v < t$ and $u < w$, and $\ll$ to be a "before" relation (t, u) *before* (v, w) if $u < v$. Hamblin (1969) and Allen (1983) showed that these suffice to define the 13 possible relations between intervals, which also include the four below, all the converses, and identity. It is an easy exercise to list the 5 possible point-interval relations (Vilain 1982).

- (t, u) *overlaps* (v, w) if $t < v < u < w$.
- (t, u) *meets* (v, w) if $u = v$.
- (t, u) *begins* (v, w) if $t = v$ and $u < w$.
- (t, u) *ends* (v, w) if $v < t$ and $u = w$.

2.2. **Interval frames to point frames.** One can start with interval frames as basic. In the open interval representation just discussed, the limiting points are given by the set of pairs of intervals which meet at that point. Formally we define $left(x) = \{(w, y) \mid w \text{ meets } y \ \& \ w \text{ meets } x\}$ provided x has intervals preceding it (and $-\infty$ otherwise), and $right(x)$ similarly. The points are ordered by the transitive closure of the relation defined by $left(x) \lhd right(x)$.

For a general interval frame $\mathcal{I} = (I, \sqsubseteq, \ll)$ where no atomicity condition holds, we resort to a standard model-theoretic construction: a *filter* F over $\mathcal{I}$ is a subset of I which is closed under $\sqsubseteq$ and finite O's. That is, if $x \in F$ and $x \sqsubseteq y$ then $y \in F$, and if x, y are in F, so is $glb(x, y)$. The point frame $Pt^+(\mathcal{I}) = (T, <)$ is now defined by letting T be the set of maximal filters over $\mathcal{I}$ and $F < G$ iff for some x in F, y in G, $x \ll y$.

An interval frame is said to be free if it satisfies:

- If for every $z \sqsubseteq x$, it is the case that zOy, then $x \sqsubseteq y$.
- If $x \not\ll y$ then for some $x_1 \sqsubseteq x, y_1 \sqsubseteq y$, it is the case that for all $x_0 \sqsubseteq x_1, y_0 \sqsubseteq y_1$, $x_0 \not\ll y_0$.

Proposition 2 ([Ben1]). *For free interval frames $\mathcal{I}$, the map $x \mapsto \{F \mid x \in F \in T\}$ is an isomorphism between $\mathcal{I}$ and $Int(Pt^+(\mathcal{I}))$.*

The adjoint map $Pt^+(Int(\mathcal{T}))$ yields isomorphism for the integers, but for $\mathcal{T} = \mathbf{Q}$, it yields a structure isomorphic to the reals.

2.3. **Point models to interval models.** Now let us try and lift these results from frames to models. Given a point model $(\mathcal{T}, TV)$, how do we define a valuation on $Int(\mathcal{T})$ which will preserve satisfaction of formulas?

The difficulty is with an interval x (recall that this is a convex subset) where some points in it are assigned a proposition p by TV and some are not. One possibility is to let $Int(TV)(x) = \{p \mid \forall t \in x : p \in TV(t)\}$. That is, our propositions are *hereditary*: if they hold during an interval, then they also hold during every subinterval. For example, "I am swimming in the water" is usually hereditary (I don't come out of the water in the middle of my swim), whereas "I am reading a book" is usually not (I don't read continuously, I might even go for a swim inbetween).

The difficulty now lies with negation. We resolve this by giving negation a "strict" or "intuitionistic" reading (Fitting 1969):

$M, x \models \neg' \alpha$ if for every $w \sqsubseteq x$, $M, w \not\models \alpha$. (This corresponds to $\Box_{down} \neg \alpha$ in the "classical" reading.)

Conjunction is unchanged, but when lifting further, we need

$M, x \models \alpha_1 \vee' \alpha_2$ if for every $w \sqsubseteq x$, there is $v \sqsubseteq w$ such that $M, v \models \alpha_1$ or $M, v \models \alpha_2$.

And similarly for $\mathsf{F}'\alpha$ ($\Box_{down} \Diamond_{down} \mathsf{F}\alpha$) and $\mathsf{P}'\alpha$ ($\Box_{down} \Diamond_{down} \mathsf{P}\alpha$).

Proposition 3 ([Ben1]). *For point frames $\mathcal{T}$, $(Int(\mathcal{T}), Int(TV)), x \models \alpha'$ iff $(\mathcal{T}, TV), t \models \alpha$ for all t in x.*

Pandya (1995) uses an interval logic with point valuations (the *duration calculus* of Zhou, Hoare and Ravn 1991) to this effect. Here is the basic idea for the language we have at hand:

Point formulas $\alpha ::= p \in Prop \mid \neg \alpha \mid \alpha_1 \vee \alpha_2$
Interval formulas $\varphi ::= \Diamond_{down} \alpha \mid \neg \varphi \mid \varphi_1 \vee \varphi_2 \mid \mathsf{P}\varphi \mid \mathsf{F}\varphi \mid \Diamond_{down} \varphi \mid \Diamond^{up} \varphi$

Pandya shows decidability of validity over dense models. Because of the point valuations, one is really working over interval models generated from point ones, which we saw have atomic frames.

2.4. **Interval models to point models.** If we start with an interval model $(\mathcal{I}, IV)$, we are faced with the question of defining a valuation $Pt^+(IV)$ for the model $Pt^+(\mathcal{I})$. Once again, the problem case is a "disagreeing" filter, for instance, containing intervals $x_1 \sqsupseteq x_2 \sqsupseteq \ldots$ with odd x_i satisfying a proposition p and even x_i not satisfying it.

The solution again lies in intuitionistic semantics (Fitting 1969): a generic filter on $M = (\mathcal{I}, IV)$ is a filter F on $\mathcal{I}$ such that for each formula α in the "point language", there is an interval x in F such that either $M, x \models \alpha$ or $M, x \models \neg' \alpha$. A generic filter can be built for any interval using a Lindenbaum construction.

We now define $Pt^*(M)$ as follows. Each generic filter on M is a point, with $G < H$ if there is some x in G, y in H such that $x \ll y$ (as before), and $Pt^*(IV)(G) = \{p \mid \exists x \in G : p \in IV(x)\}$.

When lifting this to formulas, we again have to go to the intuitionistic reading. The result therefore holds for formulas which represent hereditary properties:

Proposition 4 ([Ben1])**.** *For interval frames $\mathcal{I}$, $(\mathcal{I}, IV), x \models \alpha'$ iff $(Pt^*(\mathcal{I}), Pt^*(IV)), G \models \alpha'$ for all G containing x.*

3. Discussion

Getting back to decidability issues, we now see that if we restrict to atomic interval frames, or to hereditary properties of intervals, decidability can be obtained by translating into the point world. Another result in this vein is by Rabinovich (1998), who shows decidability over the real line for interval logic over so-called *finitely varying* models, where a proposition can only change value finitely many times during any finite interval. Of course, Halpern and Shoham's (1991) proof makes use of models and formulas which violates such properties.

Galton (1984) takes a different approach motivated by considerations of modelling natural language. Propositions p have an interval valuation.[4] Thus my writing this article will be a proposition true for the interval starting from when I started writing it to when I will stop writing it. It will not be true at any proper subinterval.

However formulas are only evaluated at points, the basic formulas $\mathsf{Perf}\, p, \mathsf{Prog}\, p, \mathsf{Pros}\, p$ translating intervals to points. They assert at a time point t that the proposition ("interval") p *has happened*, *is in progress*, or *will happen*, respectively. Thus at the present point of time, Prog "I write this article" holds. The formulas are now closed under boolean operations and point modalities $\mathsf{P}\alpha, \mathsf{F}\alpha$. Completeness is obtained by adding a few axioms to the point logic system. For instance, one axiom says that if "I will have written this article" (future perfect) is to be true, then either I have written this article or I am writing it or I will write it: $\mathsf{F}(\mathsf{Perf}\, p) \supset \mathsf{Perf}\, p \vee \mathsf{Prog}\, p \vee \mathsf{Pros}\, p$.

Galton also suggests interesting point-to-interval translations: $\mathsf{Ingr}\, \alpha$ (if α is "it is raining", then $\mathsf{Ingr}\, \alpha$ holds in an interval where it begins to rain), $\mathsf{Egr}\, \alpha$ (holds in an interval where α stops happening) and $\mathsf{Po}\, \alpha$ (if α is "I am swimming", then $\mathsf{Po}\, \alpha$ holds in the interval during which I take a swim). Formally we can build an expressive logic, for which no results seem to be known.

Point formulas $\alpha ::= \mathsf{Perf}\, \varphi \mid \mathsf{Prog}\, \varphi \mid \mathsf{Pros}\, \varphi \mid \neg\alpha \mid \alpha_1 \vee \alpha_2 \mid \mathsf{P}\alpha \mid \mathsf{F}\alpha$
Interval formulas $\varphi ::= p \in Prop \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \mathsf{Ingr}\, \alpha \mid \mathsf{Egr}\, \alpha \mid \mathsf{Po}\, \alpha$

References

[All] J.F. Allen. Maintaining knowledge about temporal intervals, *Commun. ACM* 26,11 (Nov 1983) 832–843.

[Ben1] J.F.A.K. van Benthem. *The logic of time* (Reidel, 1983).

[4] Galton works with *events* and *states*, which are more sophisticated notions. Here we simplemindedly translate his ideas to intervals and points.

[Ben2] J.F.A.K. van Benthem. Temporal logic, in [GHR], 241–350.
[Fit] M.C. Fitting. *Intuitionistic logic, model theory and forcing* (North-Holland, 1969).
[GHR] D.M. Gabbay, C. Hogger and J.A. Robinson (eds), *Handbook of logic in artificial intelligence and logic programming* IV (Oxford, 1995).
[Gal1] A.P. Galton. *The logic of aspect* (Oxford, 1984).
[Gal2] A.P. Galton. Time and change for AI, in [GHR], 175–240.
[HS] J.Y. Halpern and Y. Shoham. A propositional modal logic of time intervals, *J. ACM* 38,4 (Oct 1991) 935–962.
[Ham] C.L. Hamblin. Starting and stopping, *The Monist* 53 (1969) 410–425.
[Pan] P.K. Pandya. Some extensions to propositional mean-value calculus: expressiveness and decidability, *Proc. CSL*, *LNCS* 1092 (1995) 434–451.
[Rab] A. Rabinovich. On the decidability of continuous time specification formalisms, *J. Logic Comput.* 8,5 (1998) 669–678.
[Vil] M.B. Vilain. A system for reasoning about time, *Proc. AAAI* (1982) 197–201.
[ZHR] Zhou C., C.A.R. Hoare and A. Ravn. A calculus of durations, *Inform. Proc. Lett.* 40,5 (Dec 1991) 269–276.

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