Quadrics and vector bundles

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Quadrics

$\mathbb{C} =$ Complex numbers.
$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$, ($n$-fold product).
Let $x_1, \cdots, x_n$ be coordinates in $\mathbb{C}^n$.
A **quadric** in $\mathbb{C}^n$ is defined by a homogeneous equation of degree 2 in $x_1, \cdots, x_n$.
Simple examples:
(1) $n = 2$: $x_1^2 - x_2^2 = 0$ (a pair of lines).
(2) $n = 3$: $x_1^2 - 4x_2x_3 = 0$.
For $x_3 = a, a \in \mathbb{C}, a \neq 0$, this becomes
$x_1^2 = 4ax_2$, a parabola.

Pencil of Quadrics

We are interested in pairs of quadrics in $\mathbb{C}^n$ of the form

$$q_1 = \sum_{i=1}^{n} X_i^2 \text{ and } q_2 = \sum_{i=1}^{n} w_i X_i^2,$$

with w_i distinct scalars.
A pencil of quadrics in $\mathbb{C}^n$ is a family of quadratic forms parametrized by $(a, b) \in \mathbb{C}^2, (a, b) \neq (0, 0)$ (i.e. there is a quadric associated to each such (a, b)).
The pair of quadrics q_1, q_2 determines the pencil
$bq_2 - aq_1, (a, b) \in \mathbb{C}^2 - (0, 0)$.
Then $q = q_1 \cap q_2$ is the intersection of all quadrics of the pencil.

Curve associated to a pencil of quadrics

For n even, one can associate a curve to this pencil.
Define a curve X_1 by

$$X_1 : y^2 = (x - w_1)(x - w_2) \cdots (x - w_n).$$

and define the map

$$p : X_1 \to \mathbb{C}; \text{ by } (x, y) \mapsto x.$$

Fix $x \in \mathbb{C}$. If $x \neq w_i$, there are two values y_1, y_2 such that (x, y_1) and (x, y_2) satisfy the equation of X_1. Thus there are two points of X_1 mapping to x, hence this is a double cover.
For $x = w_i$, there is a unique point $(w_i, 0)$ in X_1 mapping to w_i.
The points $w_1, \cdots, w_n$ are called the ramification points.

Projective space $\mathbf{P}^n$

$\mathbb{C}^{(n+1)*} = \mathbb{C}^{n+1} \setminus \{(0, \cdots, 0)\}$
$= \{(x_0, \cdots, x_n) \mid x_i \in \mathbb{C}, x_i \neq 0 \text{ for some } i\}$.
$\mathbb{C}^*$ acts on $\mathbb{C}^{n+1} \setminus \{(0, \cdots, 0)\}$ by the following:
$\lambda \in \mathbb{C}^*, \lambda(x_0, \cdots, x_n) = (\lambda x_0, \lambda x_1, \cdots, \lambda x_n)$.

$$\mathbf{P}^n := \mathbb{C}^{n+1} \setminus \{(0, \cdots, 0)\} / \mathbb{C}^*.$$

An element in this quotient is an equivalence class of elements of $\mathbb{C}^{n+1} \setminus \{(0, \cdots, 0)\}$ with class
$[(x_0, \cdots, x_{n+1})] = [(\lambda x_0, \cdots, \lambda x_n)]$ and
if $(x_0, \cdots, x_n)$ and $(y_0, \cdots, y_n)$ are in same class then there is a $\lambda \in \mathbb{C}^*$ such that $(y_0, \cdots, y_n) = (\lambda x_0, \cdots, \lambda x_n)$. Each such equivalence class is a 1-dimensional subspace of $\mathbb{C}^{n+1}$ and it determines a point of $\mathbf{P}^n$.

Projective line $\mathbf{P}^1$

$n = 0$: We have $\mathbf{P}^0 = \{$ a point $\}$.
$n = 1$: Let $(x_0, x_1) \in \mathbb{C}^2 \backslash \{(0,0)\}$.
If $x_0 = 0$, $(0, x_1) = x_1(0,1)$, $[(0, x_1)] = [(0,1)]$ corresponds to one point in the projective line.
If $x_0 \neq 0$, we can divide by x_0, and define a map

$$f : \mathbf{P}^1 - [(0,1)] \to \mathbb{C}; \quad f([(x_0, x_1)]) := \frac{x_1}{x_0}.$$

Note that if $(x_0, x_1) \mapsto \frac{x_1}{x_0} \in \mathbb{C}$, $(\lambda x_0, \lambda x_1) \mapsto \frac{\lambda x_1}{\lambda x_0} = \frac{x_1}{x_0}$.
The inverse map is given by $[(1, y)] \leftarrow y \in \mathbb{C}$.
Thus the subset of $\mathbf{P}^1$ corresponding to points with $x_0 \neq 0$ is in bijective correspondence with $\mathbb{C}$.
The extra point $(0, 1)$ may be regarded as point at ∞. Thus

$$\mathbf{P}^1 = \mathbb{C} \cup \{\infty\}.$$

Projective plane $\mathbf{P}^2$

$n = 2$: Let $(x_0, x_1, x_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$.
If one coordinate, say $x_1 = 0$, then $x_0 x_2 \neq 0$ and $[(x_0, 0, x_2)] \leftrightarrow [(x_0, x_2)] \in \mathbf{P}^1$. Thus the subset of $\mathbf{P}^2$ with $x_1 = 0$ can be identified to $\mathbf{P}^1$.
If $x_1 \neq 0$, we get a bijection

$$S_2 := \{[(x_0, x_1, x_2)] \in \mathbf{P}^2 \mid x_1 \neq 0\} \longleftrightarrow \mathbb{C}^2 \text{ given by}$$

$$[(x_0, x_1, x_2)] \mapsto (\frac{x_0}{x_1}, \frac{x_2}{x_1}) \text{ and}$$

$$[(y_0, 1, y_2)] \leftarrow (y_0, y_2).$$

Thus the subset S_2 of $\mathbf{P}^2$ can be idenified to $\mathbb{C}^2$. We have

$$\mathbf{P}^2 = \mathbb{C}^2 \cup \mathbf{P}^1$$

where $\mathbf{P}^1$ a line at ∞.

Let $q(x_1, \cdots, x_n) = 0$ be a quadric in $\mathbb{C}^n$.
For $\lambda \in \mathbb{C}^*$, $q(\lambda x_1, \cdots, \lambda x_n) = \lambda^2 q(x_1, \cdots, x_n) = 0$. Hence we may write $q([x_1, \cdots, x_n]) = 0$ for $[x_1, \cdots, x_n] \in \mathbf{P}^{n-1}$. Thus a quadric q in $\mathbb{C}^n$ defines a quadric Q in $\mathbf{P}^{n-1}$.
We go back to our pencil of quadrics

$$q_1 = \sum_{i=1}^{n} X_i^2; \quad q_2 = \sum_{i=1}^{n} w_i X_i^2,$$

with w_i distinct scalars.
Let Q_1, Q_2 be the quadrics in $\mathbf{P}^{n-1}$ determined by $q_1 = 0, q_2 = 0$.

hyperelliptic curve associated to the pencil

A projective curve X is a curve in some projective space such that X is defined by finitely homogeneous polynomials.
Recall that we associated to the pencil a curve X_1.
One can obtain a projective curve X by adding two points at infinity to X_1 and map them to the point at ∞ in $\mathbf{P}^1$ (a nontrivial construction).
Thus the pencil determines

$$\text{a double cover } \; p : X \to \mathbf{P}^1$$

ramified over $w_1, \cdots, w_n$,
such a curve is called an **hyperellptic curve**.
The map $i : X \to X$ defined by

$$i(x, y) \; = \; (x, -y)$$

satisfies $i \circ i = Id_X$.
It is called the **hyperelliptic involution** on X.

Vector bundles

Roughly speaking, a vector bundle E on X associates to every point x in X an r -dimensional vector space E_x, called the fibre of E at x. $E = \cup_{x \in X} E_x$. There is a map

$$pr : E \to X \text{ defined by } pr(v) = x \text{ for } v \in E_x.$$

Moreover E is locally trivial : For every $x \in X$, there is an open set $U \subset X$ such that $\cup_{x \in U} E_x$ can be identified to the product $U \times \mathbb{C}^n$.
A vector bundle is called a line bundle if $r = 1$.
Associated to E are:

- Rank of $E = r =$ dim. E_x,
- Determinant of E, a line bundle
- Degree of E, an integer.

Examples

1. **Trivial bundle of rank r** is the vector bundle $X \times \mathbb{C}^r$.
2. **Tangent bundle** For $x \in X$, the tangent to X at x spans a 1-dimensional vector space T_x.
   Let T be the vector bundle which associates T_x to x.
   This is a line bundle called the tangent bundle of X.
3. **Direct sums:** The direct sum $E \oplus F$ of vector bundles E and F is the vector bundle obtained by associating to x the direct sum of vector spaces $E_x \oplus F_x$.
   Recall that $E_x \oplus F_x$ consists of pairs $(v, w), v \in E_x, w \in F_x$.
   One can similarly define direct sum of finitely many vector bundles.

Classification of Vector Bundles

Vector bundles on $\mathbf{P}^1$: For $n = 2$, one has $X = \mathbf{P}^1$.
For each degree d, there is a unique line bundle of degree d on $\mathbf{P}^1$.
As d varies over integers, the line bundles on $\mathbf{P}^1$ form a group isomorphic to the group of integers.
A vector bundle on $\mathbf{P}^1$ is a direct sum of line bundles (Grothendick).
Vector bundles on an elliptic curve. For $n = 4$, the curve X is called an elliptic curve. $\{$ Line bundles on $X\} \longleftrightarrow X$.
Vector bundles on an elliptic curve were studied and classified completely by M. Atiyah.
For $n \geq 4$, fix rank n and degree d. Even then it is impossible to find a geometric object which parametrises all vector bundles on X. So we restrict to 'good' vector bundles called semistable vector bundles.
Definition . A vector bundle E of rank n and degree d is called stable (respectively semistable) if for every proper subbundle F of E, we have

$$\textit{degree } F / \textit{ rank } F < (\textit{resp.} \leq) \; d/n.$$

Moduli spaces

Mumford and Seshadri showed that there exist geometric objects (curves, surfaces etc.) whose points are in bijective correspondence with stable vector bundles (or certain classes of semistable vector bundles). They are called **Moduli spaces** of stable (semistable) vector bundles.

$U(n, d) =$ moduli space of semistable vector bundles of rank n, degree d on X.

$U_L(n, d) =$ moduli space of semistable vector bundles of rank n, degree d, with a fixed determinant L on X.

These moduli spaces were constructed by complicated constructions.

Question: What do these moduli spaces look like?

Explicit descriptions of moduli spaces

Theorem [P.E. Newstead, S. Ramanan, M.S. Narasimhan]

- $n = 4$: $U(2,1) \cong X$, $U(2,0) \cong \mathbf{P}^1$
- $n = 6$:
  (1) $U_L(2,1) \cong$ Intersection of 2 quadrics in $\mathbf{P}^5$ given by

$$q_1 = \sum_{j=1}^{6} X_j^2, q_2 = \sum_{j=1}^{6} w_j X_j^2,$$

  w_j being all distinct scalars.
  (2) $U_L(2,0) \cong \mathbf{P}^3$

Definition 1

Projective varieties are subsets of $\mathbf{P}^n$ which are the zero sets of finitely many homogeneous polynomials.

Isotropic spaces

Definition 2

A subspace V contained in a quadric q in $\mathbb{C}^n$ is called an isotropic subspace for q.

Maximum isotropic subspaces for q_1 (or q_2) have dimension equal to the integral part of $n/2$.
For n even, the maximum isotropic spaces form two (isomorphic) systems.

Definition 3

Rank of a quadric: If V is not contained in q, then $q\,|_V := V \cap \{q = 0\}$ is a quadric in V. Choosing a basis of V, let $y_1, \cdots, y_m$ be coordinates in $V \cong \mathbb{C}^m$. Then $q\,|_V$ can be written as $\sum_{i,j} A_{i,j}\, y_i y_j$. The rank of the matrix A with entries $A_{i,j}$ is called the rank of $q\,|_V$.

Recall that the rank of A is the minimal integer r such that A has a non-vanishing $r \times r$-minor.

Generalization

Theorem [Usha Bhosle- S.Ramanan].

1. Let X be a smooth hyperelliptic curve, $n = 2g + 2 \geq 6$. Then $U_L(2,1) \cong$ the variety of $(g-1)$ -dimensional subspaces of $\mathbb{C}^{2g+2}$ which are contained in the intersection of two quadrics
$$q_1 = \sum_{j=1}^{2g+2} X_j^2, \; q_2 = \sum_{j=1}^{2g+2} w_j X_j^2,$$
w_j being mutually distinct scalars.
2. $R =$ the variety of $g+1$ dimensional subspaces V of $\mathbb{C}^{2g+2}$ which satisfy the following conditions:
(i) V is contained in q_1 and belongs to a fixed system of maximum isotropic spaces. (ii) $q_2 \mid V$ has rank ≤ 4. Then
$$U_L(2,0)/i \cong R,$$
$i =$ canonical involution induced by hyperelliptic involution.

Applications to Mathematics

- Quadrics and spaces associated to quadrics have been studied extensively in classical geometry. This wealth of information gave information about various properties of $U(n,d)$ and $U_L(n,d)$ over hyperelliptic curves.
  This lead to good conjectures for these spaces over any curve too. For example, the group of line bundles on $\mathbf{P}^3$ or on the intersection of quadrics in $\mathbf{P}^5$ is isomorphic to the group of integers. It was conjectured that the group of line bundles on U_L is isomorphic to the group of integers over any smooth curve, conjecture was proved to be true [Seshadri, Ramanan, Drezet-Narasimhan].
- The results generalise to orthogonal bundles, symplectic bundles, spin bundles etc. on X. They also generalize to nodal curves i.e. in case some distinct pairs of w_i coincide.

Application to Physics

Since the (multiplicative) group of line bundles on $U_L = U_L(r, d)$ is isomorphic to the group of integers, every line bundle on U_L is of form θ^k, where θ is the positive generator of this group.
A section of θ^k is a map $s : U_L \to \theta^k$ such that for $u \in U_L, s(u) \in (\theta^k)_u$.

$$H^0(\theta^k) := \text{ the space of sections of } \theta^k.$$

Conformal field theory in theoretical physics associates to each curve X a vector space called the space of conformal blocks, this space can be identified to $H^0(\theta^k)$.
Verlinde conjecture gave explicit formulas for the dimensions of the conformal blocks.
[Szenes] : Verlinde formulas true for rank $r = 2$.
His proof crucially uses our explicit descriptions of moduli spaces.