Polynomials in knot theory

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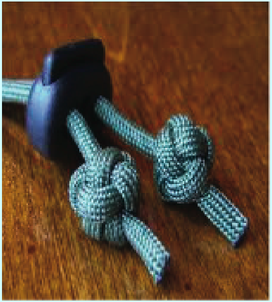
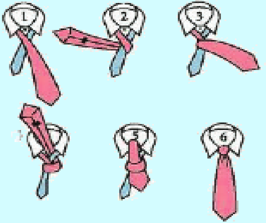
Knots in the real world



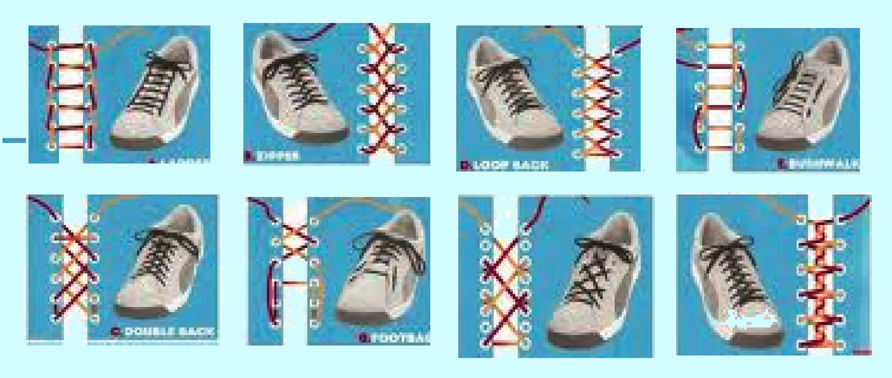
Knots in shoelaces



Knots in ties

Knots in the hair plats

The fact that you can tie your shoelaces in several ways has inspired mathematicians to develop a deep subject known as knot theory.

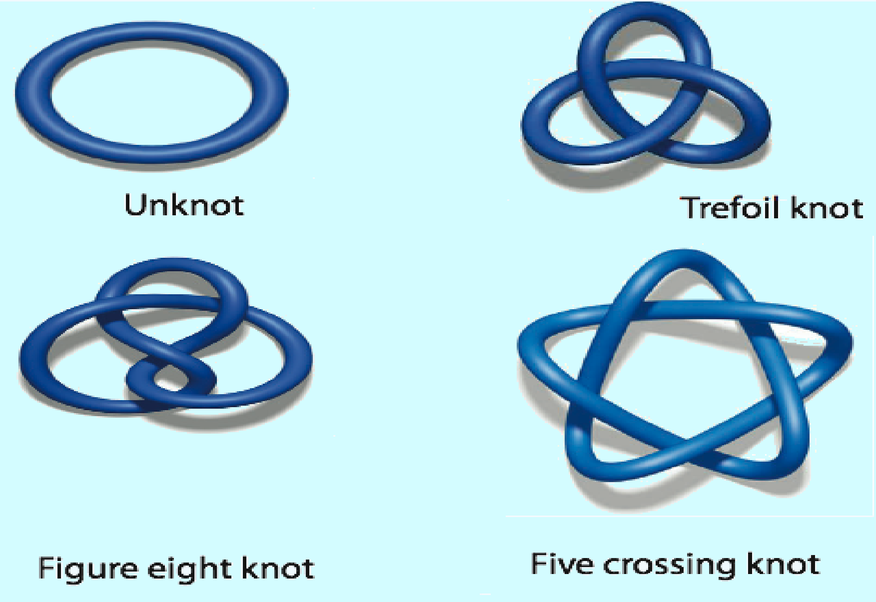


mathematicians treat knots as a mathematical objects and do all sorts of maths on it.

Mathematical definition of a knot

A knot is a simple closed curve inside the three dimensional space $\mathbb{R}^3$. If the simple closed curve is drawn without a strand passing under another strand, it is called a trivial knot or an unknot.

Examples



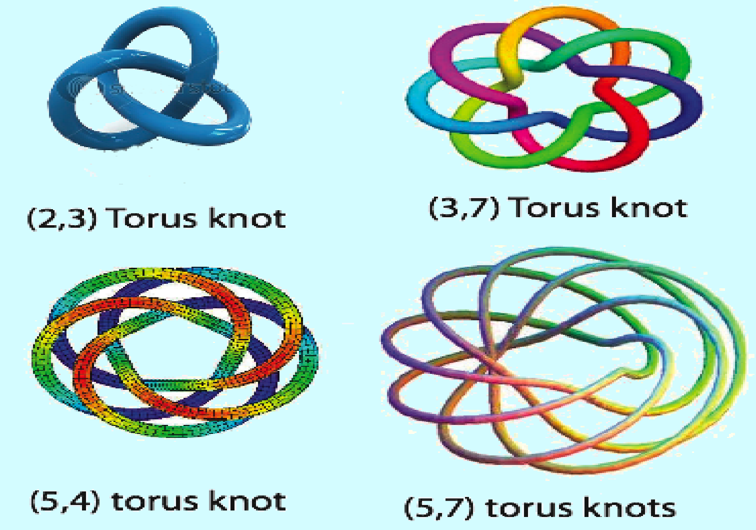
A simple closed curve in $\mathbb{R}^3$ can be regarded as an image of a smooth embedding of unit circle S^1 in $\mathbb{R}^3$. Since $\mathbb{R}^3$ can be thought of as a subset of unit three sphere S^3, we can regard a knot as a smooth embedding of S^1 in S^3.

There can be infinitely many smooth embedding of S^1 in S^3 and image of S^1 under each one of them is diffeomorphic to S^1. Thus, we need to find some criteria on which we can classify knots.

Knot theory deals with what is known as placement problem.

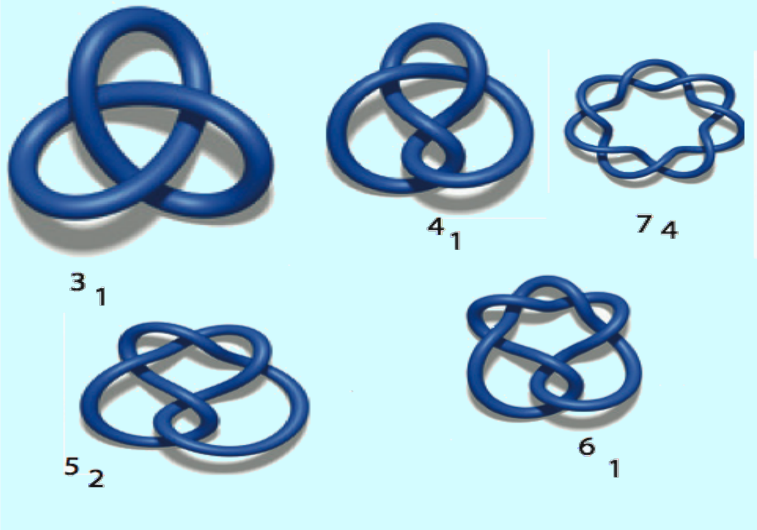
Some important classes of knots

Torus knots of type (p, q)

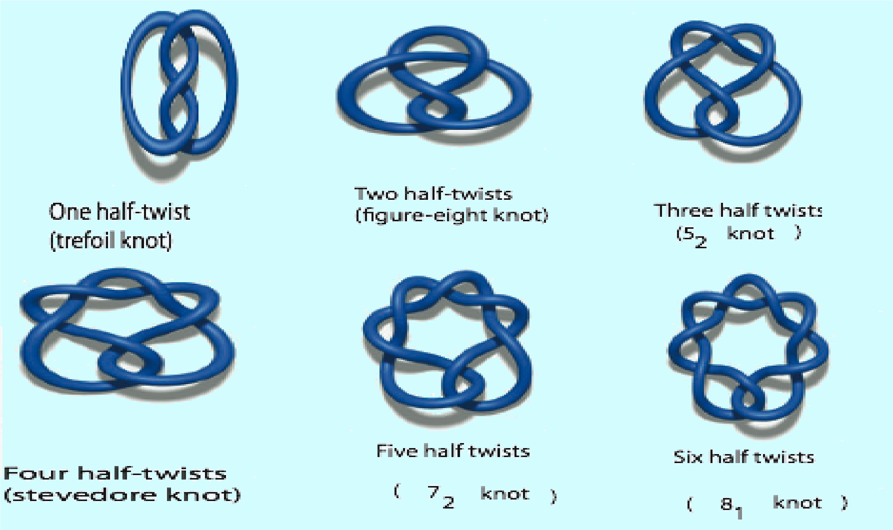


Some important classes of knots

2-bridge knots or rational knot



Twist knots

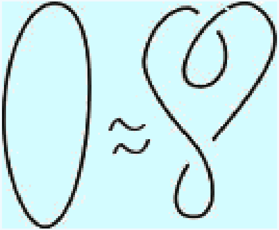


One half-twist
(trefoil knot)
Two half-twists
(figure-eight knot)
Three half twists
(5_2 knot)
Four half-twists
(stevedore knot)
Five half twists
(7_2 knot)
Six half twists
(8_1 knot)

Knot equivalence

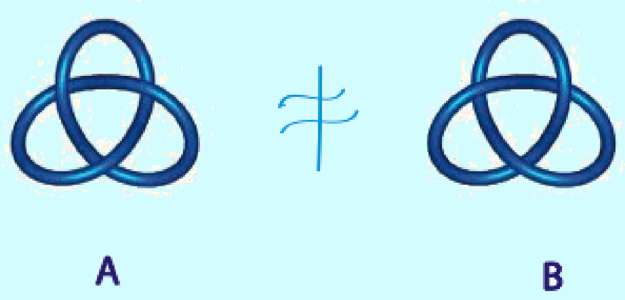
In a layman's language if we can change a knot into another knot without cutting or tearing it, then we say that these two knots are equivalent.

For examples, these two knots drawn below are equivalent and it is easy to prove it.



Knot equivalence

One can try hard but cannot convert the knot A into the knot B drawn below



However, it is difficult to prove this.

Formal definition of equivalence of knots

A knot K_1, defined by the embedding φ_1, is said to be ambient isotopic to knot K_2 defined by the embedding φ_2 if there exists an orientation preserving diffeomorphism $h : S^3 \longrightarrow S^3$ such that $h \circ \varphi_1 = \varphi_2$. Being isotopic is an equivalence relation in the set of all knots.

This matches with our intuitive idea of knot equivalence. However, it is very difficult to prove knot equivalence using this definition.

Thus, we need to find some properties or the structures or quantities that are preserved under this equivalence. These are known as knot invariants.

The main objective in knot theory is to invent more and more powerful invariants.

An immediate invariant that comes to mind is the topological space $S^3 \setminus K$, the complement of knots. This has been known that two knots are ambient isotopic if and only if their complements are isotopic.

Reference: Knots are determined by their complements, Author(s): C. McA. Gordon; J. Luecke Journal: Bull. Amer. Math. Soc. 20 (1989), 83-87.

But proving two smooth manifolds are diffeomorphic or not is also hard.

Then one may consider the knot group which is $\Pi_1(S^3 \setminus K)$. It is easy to prove that if two knots are equivalent then their knot groups are isomorphic.

Knot group of a knot is able to detect when a knot K is not equivalent to an unknot due to the following

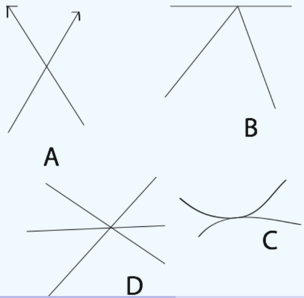
Theorem: A knot K is an unknot if and only if its knot group is infinite cyclic.

However, a knot and its mirror image have isomorphic knot groups. Thus it cannot differentiate a knot from its mirror image.

Thus we need stronger knot invariants.

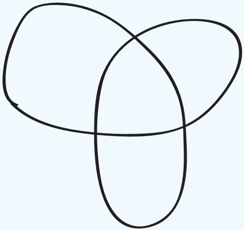
It is difficult to work with knots as embeddings into three dimensional space. Life is much simpler if we can project it into a nice plane and capture all the information.

The choice of plane should be such that the projected image is a generic immersion, i.e., consists of only transverse double points, no tangential intersections, no cusps are allowed. For instance in the figure drawn below a projected image may have a self intersection as in the diagram A and should not have diagrams B, C and D

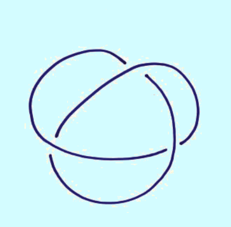


Such a projection always exists. In fact the set of such projections is an open and dense set in the space of all projections. This type of projection is known as a regular projection.

A typical regular projection of trefoil knot looks as shown below

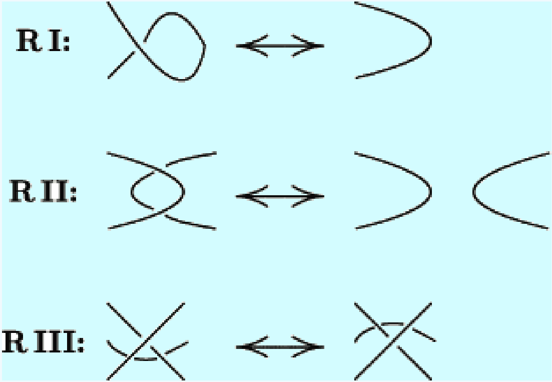


By looking at a knot projection we cannot understand which embedding it comes from. Thus at each crossing point we provide a choice of over/under crossing and the resulting picture is called a knot diagram as shown below



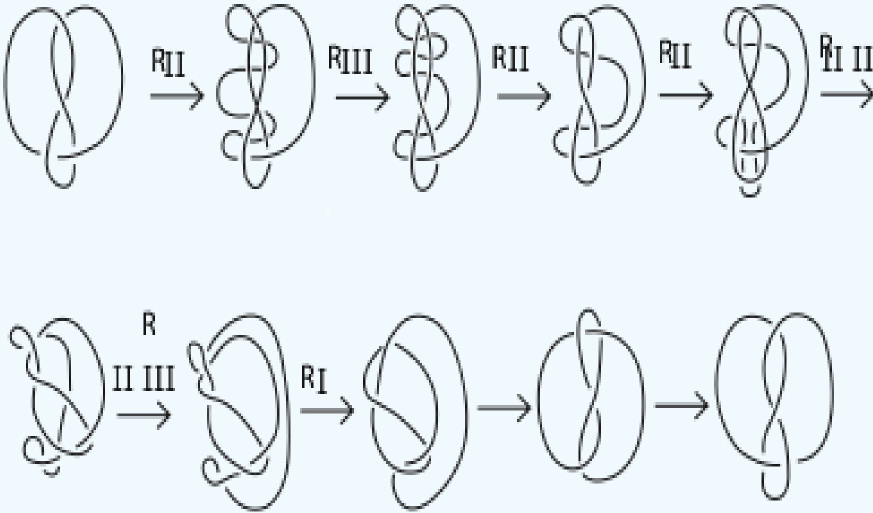
Knot diagrams are a sufficient data to tell you about the knot, except that there may be several knot diagrams for a particular knot. We need an equivalence at the diagram level!

Reidemeister's Moves



Theorem: Two knot diagrams represent isotopic knots if and only if one can be transformed into the other by a sequence of finitely many Reidemeister Moves.

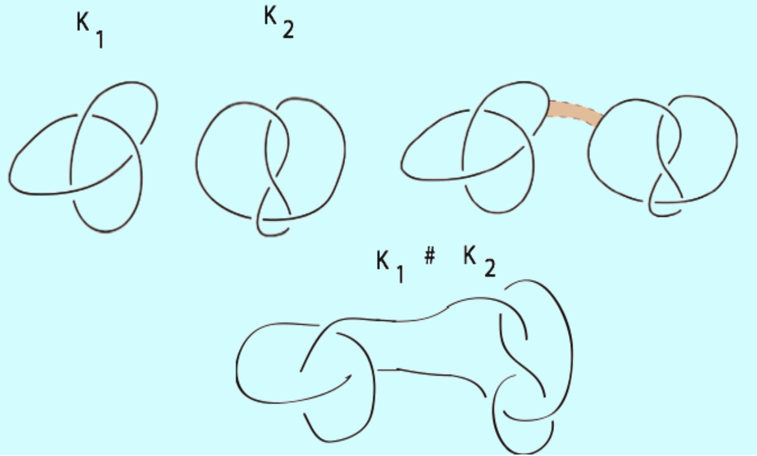
Demonstrating Reidemeister's Moves



Equivalence of figure eight knot and its mirror image

There are many mathematical structures that are associated to a knot diagram. If they remain invariant under all three Reidemeister Moves then they serve as a knot invariant. Surprisingly many knot invariants are polynomials.

Connected Sum of knots:



Why many knot invariants often are polynomials ?

A knot that is connected sum of two non trivial knots is a composite knot. A knot K is a prime knot if it is not a composite knot.

The set G of isotopy classes of knots form a commutative monoid under the connected sum operation.

Knots can be treated as polynomials

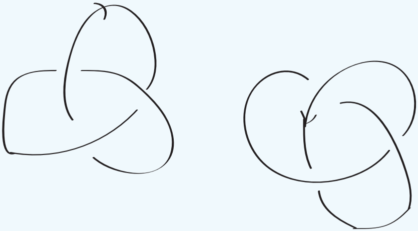
Let $\mathcal{K}(\mathcal{G})$ denote the formal linear combination of elements of G. Then $\mathcal{K}(\mathcal{G})$ has a structure of a commutative ring with respect to ordinary addition and connected sum. It can be shown that $\mathcal{K}(\mathcal{G})$ is a free ring with prime knots as generators.

From Algebra we know that polynomial rings are free rings. Thus, in some sense, knots can be regarded as polynomials. Also, we can find polynomial invariants easily.

The polynomials can be calculated by a simple, albeit long, algorithm. A few terms need to be defined in order for the polynomials to be possible to calculate. The first such term is orientation.

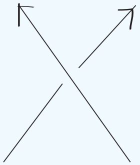
Orientation and writhe

For any knot, we can define an orientation or direction of the knot. To do this, we simply put an arrow somewhere on the knot and state that that arrow specifies the direction. For example



Orientation and writhe

The writhe $w(D)$ of a knot diagram D is defined to be the sum of all the crossings of the knot. We give each crossing a number: either -1 or $+1$. Here is how we can tell whether a crossing gets $a+1$ or $a-1$ number:



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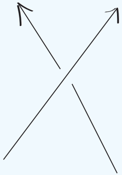


+ 1

skein relation

K_+ K_- K_0

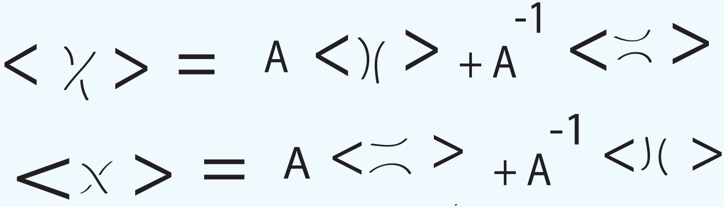
smoothing

Alexander Polynomial

The Alexander polynomial of a knot was the first polynomial invariant discovered. It was discovered in 1928 by J. W. Alexander, and until the 1980s, it was the only polynomial invariant known. Alexander used the determinant of a matrix to calculate the Alexander polynomial of a knot. But this can also be done using the skein relations. There are three rules that are used when calculating the Alexander polynomial of a knot:

1. $\Delta(Unknot) = 1.$
2. $\Delta(2\ \ Unknots) = 0.$
3. $\Delta(K_+) - \Delta(K_-) + (t^{\frac{1}{2}} - t^{\frac{-1}{2}})\Delta(K_0) = 0.$

$$\langle \bigcirc \rangle = 1$$



$$\langle L \cup \bigcirc \rangle = (-A^{2} - A^{-2}) \langle L \rangle$$

Bracket polynomial is not a knot invariant because we have

Kauffman Polynomial

Let K be a knot and D is a diagram of K. Then the Kauffman polynomial $F(K)$ is defined as

$$F(K) = (-A)^{-w(D)} < D >$$

Jones Polynomial

$$V_K(t) = F(K)(t^{\frac{-1}{4}}).$$

HOMFLY Polynomial

The HOMFLY polynomial is a generalization of the Alexander and Jones polynomials. Instead of being a polynomial in one variable as the other two are, it is a polynomial in two variables. It was discovered in 1985 by J. Hoste, A. Ocneanu, K. C. Millett, P. J. Freyd, W. B. R. Lickorish, and D. N. Yetter. For this polynomial, there are only two rules:

1. $P(Unknot) = 1$
2. $l\, P(K_+) + l^{-1} P(K_-) + mP(K_0) = 0$

There are many more polynomial invariants that are variations of this two variable polynomial.

Polynomial Parametrization of knots

Knots in S^3 can be realized as one point compactification of embeddings of $\mathbb{R}$ in $\mathbb{R}^3$. Later are called open knots or non-compact knots.

Two non-compact knots $(\widetilde{\phi_1} : \mathbb{R} \longrightarrow \mathbb{R}^3)$ and $(\widetilde{\phi_2} : \mathbb{R} \longrightarrow \mathbb{R}^3)$ are said to be equivalent if their extensions ϕ_1 and ϕ_2 from S^1 to S^3 are ambient isotopic. An equivalence class of a non-compact knot is a knot-type.

An embedding of $\mathbb{R}$ in $\mathbb{R}^3$ is given be $t \mapsto (f(t), g(t), h(t))$ where $f(t), g(t)$ and $h(t)$ are real polynomials in one variable, is called a polynomial knot.

Theorem:[Shastri, 1990] Every non compact knot is equivalent to some polynomial knot.

Theorem: Two polynomial embeddings $\phi_0, \ \phi_1 : \mathbb{R} \hookrightarrow \mathbb{R}^3$ representing the same knot-type are polynomially isotopic.

By polynomially isotopic we mean that there exists $\{P_t : \mathbb{R} \hookrightarrow \mathbb{R}^3 | \ t \in [0,1]\}$, a one parameter family of polynomial embeddings, such that $P_0 = \phi_0$ and $P_1 = \phi_1$.

Both these theorems were proved using *Weierstrass' approximation*. Thus the nature and the degrees of the defining polynomials cannot be estimated.

Minimal Degree Sequence of knots

A triple $(l, m, n) \in \mathbb{N}^3$ is said to be a degree sequence of a given knot-type K if there exists $f(t)$, $g(t)$ and $h(t)$, real polynomials, of degrees l, m and n respectively, such that the map $t \mapsto (f(t), g(t), h(t))$ is an embedding which represents the knot-type K.

A degree sequence $(l, m, n) \in \mathbb{N}^3$ for a given knot-type is said to be the minimal degree sequence if it is minimal among all degree sequences for K with respect to the lexicographic ordering in $\mathbb{N}^3$.

If (l, m, n) is a degree sequence of knots, we may always assume that $l < m < n$.

Degree Sequence of knots: Some known results

Theorem: A torus knot of type $(2, 2n+1)$ has a degree sequence $(3, 4n, 4n+1)$.

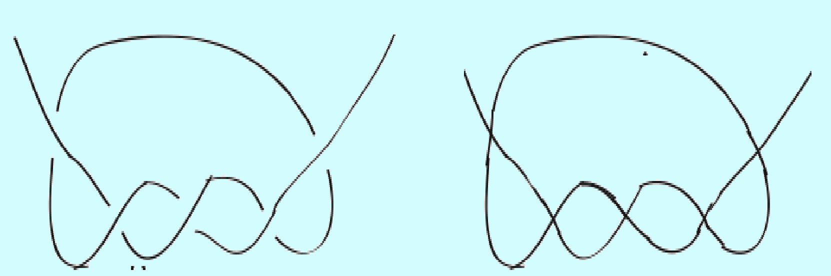
Theorem : A torus knot of type $(p, q), p < q, p > 2$ has a degree sequence $(2p-1, 2q-1, 2q)$.

It is easy to observe that these degree sequences are not the minimal degree sequence for torus knots.

Constructing a polynomial representation

In order to represent a knot-type by a polynomial embedding we require a suitable knot diagram.

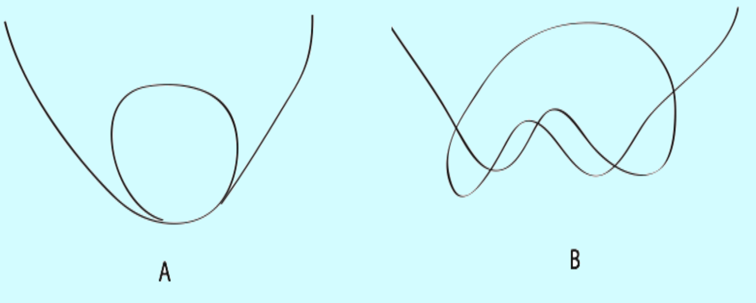
For example, a knot diagram for a torus knot of type $(2, 5)$ may be taken as the diagram A shown below. We can think of a regular projection into X,Y plane and the projection may be taken as the diagram B.



Let $f(t) = t(t^2 - 3)$ and $g(t) = t^5(t^2 - 3)^4$ Then a plane curve

$$(X(t), Y(t)) = (f(t), g(t))$$

has the image as shown in the diagram A below



We can show that $(0, 0)$ is the only real singular point.

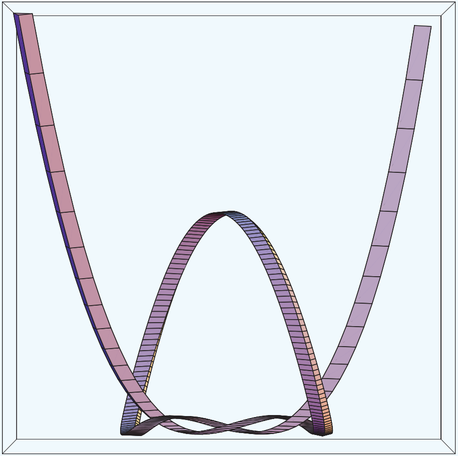
In this curve we can study the two local branches at origin and compute their intersection multiplicity at origin which turns out to be 5 in this case.

Now, using Algebraic geometry argument, we can perturb the coefficients of $g(t)$ and create all the double points in the neighborhood of $(0, 0)$. Taking $g(t) = (t^2 - 1.2)(t^2 - 2.25)(t^2 - 3.9)(t^2 - 4.85)$ we obtain the curve as in the diagram B.

Let

$$h(t) = (t^2 - 2.26311^2) * (t^2 - 2.116775^2) * (t^2 - 1.812575^2) * (t^2 - 1.26559$$

Then the image of 3D picture of the embedding $t \mapsto (f(t), g(t), h(t))$ is obtained in Mathematica as



Minimal degree sequence

Theorem: The minimal degree sequence for torus knot of type $(2,\ 2n+1)$ for $n = 3m;\ 3m+1\ \ and\ \ 3m+2$ is $(3,\ 2n+2,\ 2n+4);\ (3,\ 2n+2,\ 2n+3)$ and $(3,\ 2n+3,\ 2n+4)$ respectively.

Theorem: The minimal degree sequence for a 2-bridge knot having minimal crossing number N is given by

1. $(3,\ N+1,\ N+2)$ when $N \equiv 0\ (mod\ 3)$;
2. $(3,\ N+1,\ N+3)$ when $N \equiv 1\ (mod\ 3)$;
3. $(3, N+2, N+3)$ when $N \equiv 2\ (mod\ 3)$

Minimal degree sequence

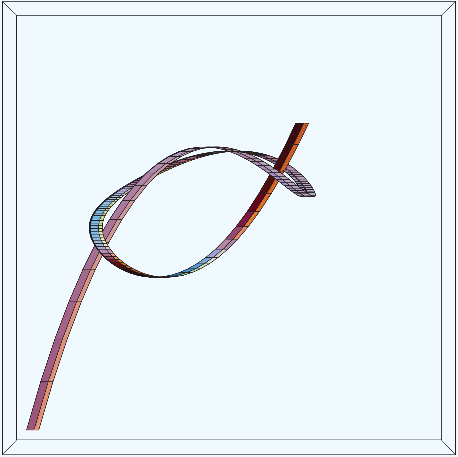
Theorem: The minimal degree sequence for a torus knot of type $(p,\ 2p-1)$, $p \geq 2$ denoted by $K_{p,2p-1}$ is given by $(2p-1,\ 2p,\ d)$, where d lies between $2p+1$ and $4p-3$.

A few Polynomial knots

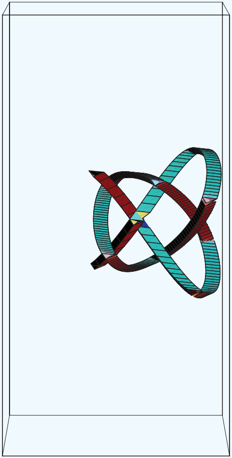
$t \mapsto (t(t-1)\times(t+1), t^2(t-1.15)\times(t+1.15), (t^2 - 1.056445^2)\times(t^2-0.644893^2)t)$

A few Polynomial knots

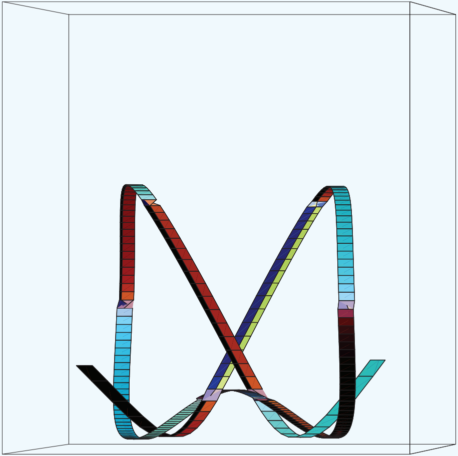
$t \mapsto (t(t-2)\times(t+2), (t-2.1)\times(t+2.1)t^3, (t^2 - 2.176385^2)\times(t^2-1.83588^2)\times(t^2-0.8956385^2)t)$



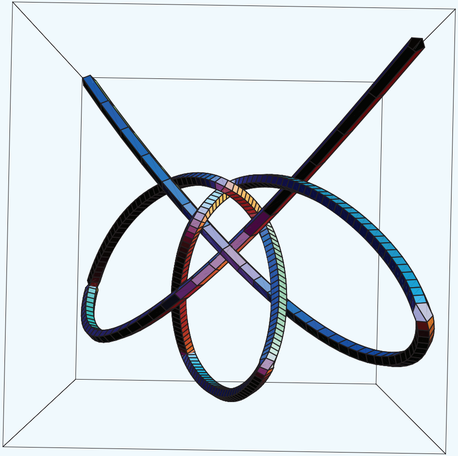
$t \mapsto (t^3 - 17t, t^7 - 0.66t^6 - 29t^5 + 43t^4 + 208t^3 - 680t^2 - 731t, (t + 4.5) \times (t + 4.1) \times (t + 3.2) \times (t + 2.3) \times (t + 0.85) \times (t - 0.2) \times (t - 1.75) \times (t - 3.65) \times (t - 4.59949))$



$t \mapsto ((t^2 - 12) \times (t^2 - 11), t(t^2 - 21) \times (t^2 - 7), (t^2 - 4.6573875^2) \times (t^2 - 4.472939^2) \times (t^2 - 3.504525^2) \times (t^2 - 2.318071^2) \times (t^2 - 1.2983325^2)t)$



$t \mapsto$
$(t(t^2-17), t^2(t^2-18)\times(t+4.7)\times(t^2-4.15)\times(t-4.7), (t^2-4.6^2)\times (t^2-4.35^2)\times(t^2-4.18^2)\times(t^2-9)\times(t^2-1.8^2)\times(t^2-0.75^2)t)$



$$t \mapsto (t^5 - 5.5 \times t^3 + 4.5 \times t, -7.8375 + 14 \times t^2 - 7.35 \times t^4 + t^6, -127.627 \times t + 563.155 \times t^3 - 909.757 \times t^5 + 672.438 \times t^7 - 236.4233 \times t^9 + 38.943 \times t^{11} - 2.4293 \times t^{13})$$