Cellular Algebras

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Introduction

- Representation theory.
- Representations and modules.
- Ordinary and modular representations.
- Semisimple and non-semisimple algebras.

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- Cellular algebras were introduced by Graham and Lehrer in 1996, and are a class of finite dimensional associative algebras defined in terms of a cell datum and three axioms.
- They obtain a general description of the irreducible representations of the cellular algebra together with the criterion for the cellular algebra to be semisimple.
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Definition

Let R be a commutative ring with unity. An associative R-algebra A is called a **cellular algebra** with cell datum (Λ, M, C, i) if the following conditions are satisfied:

(C1) Suppose that $(\Lambda, \geq)$ is a (finite) poset. Associated with each $\lambda \in \Lambda$ there is a finite set $M(\lambda)$. The algebra A has an R-basis $C^{\lambda}_{S,T}$ where (S, T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in \Lambda$.

(C2) The map i is an R-linear anti-automorphism of A with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

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(C2) The map i is an R-linear anti-automorphism of A with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $\sum_{U \in M(\lambda)} r_a(U,S)C^{\lambda}_{U,T} + r'$ where r' is a linear combination of basis elements with upper index μ strictly bigger than λ, and where the coefficients $r_a(U,S) \in R$ do not depend on T.

Example

An easy example of a cellular algebra is provided by the algebra of n by n matrices.

- Let $A = Mat_{n\times n}(R)$ and take $\Lambda = \{n\}$ and $M(n) = \{1, \cdots, n\}$.
- The set of elementary matrices $\{E_{i,j} | 1 \leq i, j \leq n\}$ give a cellular basis of A.
- For any $a \in A$, we have $aE_{i,j} = \sum_k r_a(k, i)E_{k,j}$.
- We take the usual matrix transpose as an involution of A.

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Example

- Let G be a cyclic group of order m generated by $x \in G$ and let ξ be a primitive mth root of unity in R and let $K = R[\xi]$, then RG is cellular over K.
- Λ is the poset $\{0, 1, \cdots, m-1\}$, ordered in the natural way, $M(\lambda)$ is an one-element set for each $\lambda \in \Lambda$ and $C^{\lambda}_{i,i}$ is $\prod_{j=1}^{i}(x - \xi^j)$, $j = 0, 1, \cdots, m-1$.
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Classification of simple modules

- Fix an element $\lambda \in \Lambda$. Define the right cell module C^{λ} to be the right A-module which is free as an R-module with basis $\{C_S^{\lambda} | S \in M(\lambda)\}$ and for each $a \in A$

$$aC_S^{\lambda} = \sum_{T \in M(\lambda)} r_T C_T^{\lambda}$$

where r_T is the element of R.

- There is a unique bilinear map $\langle , \rangle : C^{\lambda} \times C^{\lambda} \longrightarrow R$ such that $\langle C_S^{\lambda}, C_T^{\lambda} \rangle$, for $S, T \in M(\lambda)$, is given by

$$\langle C_S^{\lambda}, C_T^{\lambda} \rangle C_{U,V}^{\lambda} \equiv C_{U,S}^{\lambda} C_{T,V}^{\lambda} + r^{'},$$

where U and V are any elements of $M(\lambda)$ and $\langle , \rangle$ is both symmetric and associative.

- Let $radC^{\lambda} = \{x \in C^{\lambda} | \langle x, y \rangle = 0 \text{ for all } y \in C^{\lambda}\}$.
- Define $D^{\lambda} = C^{\lambda} / radC^{\lambda}$.
- Let $\Lambda_0 = \{\mu \in \Lambda | D^{\mu} \neq 0\}$ Then $\mu \in \Lambda_0$ iff the bilinear form $\langle , \rangle$ on C^{μ} is non-zero.

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Theorem

(Graham and Lehrer) Suppose that R is a field and that Λ is finite. Then $\{D^\mu | \mu \in \Lambda_0\}$ is a complete set of pairwise inequivalent irreducible A-modules.

- Suppose that $\mu \in \Lambda_0$ and $\lambda \in \Lambda$. Define $d_{\lambda\mu} = [C^\lambda : D^\mu]$ to be the decomposition number of the irreducible module D^μ in C^λ.
- The matrix $D = (d_{\lambda\mu})$, where $\lambda \in \Lambda$ and $\mu \in \Lambda_0$, is the so-called decomposition matrix of A.

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Corollary

Suppose that R is a field. Then the decomposition matrix D of A is unitriangular. That is, if $\mu \in \Lambda_0$ and $\lambda \in \Lambda$ then $d_{\mu\mu} = 1$ and $d_{\lambda\mu} \neq 0$ only if $\lambda \geq \mu$.

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Suppose that R is a field and that Λ is finite. Let C be the Cartan matrix of A. Then $C = D^t D$. In particular, the Cartan matrix of A is symmetric.

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Criterion for semisimplicity

Corollary

Suppose that R is a field. Then the following are equivalent.

(i) *A is (split) semisimple.*

(ii) $C^\lambda = D^\lambda$ *for all* $\lambda \in \Lambda$.

(iii) $radC^\lambda = 0$ *for all* $\lambda \in \Lambda$.

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Applications to Diagram algebras

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