MURRE'S CONJECTURE FOR A RATIONAL HOMOGENEOUS BUNDLE OVER A VARIETY

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Abstract. In this paper, we investigate Murre's conjectures on the structure of rational Chow groups for a rational homogeneous bundle $Z \to S$ over a smooth variety. Absolute Chow-Künneth projectors are exhibited for Z whenever S has a Chow–Künneth decomposition.

1. Introduction

Suppose X is a nonsingular projective variety of dimension n defined over the complex numbers. Let $CH^i(X) \otimes \mathbb{Q}$ be the Chow group of codimension i algebraic cycles modulo rational equivalence, with rational coefficients. Jacob Murre [Mu2], [Mu3] has made the following conjecture which leads to a filtration on the rational Chow groups:

Conjecture: The motive $h(X) := (X, \Delta_X)$ of X has a Chow-Künneth decomposition:

$$\Delta_X = \sum_{i=0}^{2n} \pi_i \in CH^n(X \times X) \otimes \mathbb{Q}$$

such that π_i are orthogonal projectors (see §2.2).

Some examples where this conjecture is verified are: curves, surfaces, a product of a curve and surface [Mu1], [Mu3], abelian varieties and abelian schemes [Sh],[De-Mu], uniruled threefolds [dA-Mü1], elliptic modular varieties [Go-Mu], [GHM2]), universal families over Picard modular surfaces [MMWYK] and finite group quotients (maybe singular) of abelian varieties [Ak-Jo], some varieties with a nef tangent bundles [Iy], open moduli spaces of smooth curves, Simpsons's *Betti moduli spaces* [Iy-Mu], universal families over some Shimura surfaces [Mi].

In [Iy], we had looked at varieties which have a nef tangent bundle. Using the structure theorems of Campana and Peternell [Ca-Pe] and Demailly-Peternell-Schneider [DPS], we know that such a variety X admits a finite étale surjective cover $X' \to X$ such that $X' \to A$ is a bundle of smooth Fano varieties over an abelian variety. Furthermore, any fibre which is a smooth Fano variety necessarily has a nef tangent bundle. It is an open question [Ca-Pe, p.170] whether such a Fano variety is a rational homogeneous variety. We showed in [Iy] that whenever the étale cover is a relative cellular variety over A or if it

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admits a relative Chow–Künneth decomposition, then X' and X have a Chow–Künneth decomposition.

In this paper, we weaken the hypothesis on the cover $X' \to A$ as above and obtain a Chow–Künneth decomposition whenever $X' \to A$ is a rational homogeneous bundle over an abelian variety. This strengthens the results in [Iy] and if the open question [Ca-Pe, p.170] is answered positively then we will obtain a Chow–Künneth decomposition for all varieties which have a nef tangent bundle.

We state the result and proofs, in a slightly more general situation.

Theorem 1.1. *Suppose Z is a rational homogeneous bundle over a smooth projective variety S. Assume that S has a Chow-Künneth decomposition. Then the motive of Z has an absolute Chow–Künneth decomposition.*

Moreover, the motive of the bundle $Z \to S$ is expressed as a sum of tensor products of summands of the motive of S with the twisted Tate motive (see Proposition 3.6).

The new ingredient in the proof is to observe that a rational homogeneous bundle is étale locally a relative cellular variety, by applying a result of Colliot-Thélène-Ojanguren [Co-Oj]. So relative Chow–Künneth projectors (in the sense of [De-Mu]) can be constructed over an étale cover. These projectors lie the subspace generated by the relative algebraic cells. The corresponding relative cohomology classes patch up since they lie in the subspace generated by the relative analytic cells. Hence the relative orthogonal projectors can be patched up as algebraic cycles over the étale site in the rational Chow groups. Hence, in this case, the relative Chow–Künneth projectors over the étale site descend to relative Chow–Künneth projectors for $Z \to S$ (see Corollary 3.5). The criterion of Gordon-Hanamura-Murre [GHM2], for obtaining absolute Chow–Künneth projectors from relative Chow–Künneth projectors can be directly applied.

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2. Preliminaries

We work over the field of complex numbers in this paper.

2.1. **Category of motives.** The category of nonsingular projective varieties over $\mathbb{C}$ will be denoted by $\mathcal{V}$. For an object X of $\mathcal{V}$, let $CH^i(X)_{\mathbb{Q}} = CH^i(X) \otimes \mathbb{Q}$ denote the rational Chow group of codimension i algebraic cycles modulo rational equivalence. Suppose $X, Y \in Ob(\mathcal{V})$ and $X = \cup X_i$ be a decomposition into connected components X_i and $d_i = \dim X_i$. Then $\mathrm{Corr}^r(X,Y) = \oplus_i CH^{d_i+r}(X_i \times Y)_{\mathbb{Q}}$ is the group of correspondences of degree r from X to Y.

We will use the standard framework of the category of Chow motives $\mathcal{M}_{rat}$ in this paper and refer to [Mu2] for details. We denote the category of motives $\mathcal{M}_{\sim}$, where

$\sim$ is any equivalence, for instance $\sim$ is homological or numerical equivalence. When S is a smooth variety, we also consider the category of relative Chow motives $CH\mathcal{M}(S)$ which was introduced in [De-Mu] and [GHM]. When $S =$ Spec $\mathbb{C}$ then the category $CH\mathcal{M}(S) = \mathcal{M}_{rat}$.

2.2. **Chow–Künneth decomposition for a variety.** Suppose X is a nonsingular projective variety over $\mathbb{C}$ of dimension n. Let $\Delta_X \subset X \times X$ be the diagonal. Consider the Künneth decomposition of Δ in the Betti Cohomology:

$$\Delta_X = \oplus_{i=0}^{2n} \pi_i^{hom}$$

where $\pi_i^{hom} \in H^{2n-i}(X) \otimes H^i(X)$.

Definition 2.1. *The motive of X is said to have Künneth decomposition if each of the classes π_i^{hom} are algebraic and are projectors, i.e., π_i^{hom} is the image of an algebraic cycle π_i under the cycle class map from the rational Chow groups to the Betti Cohomology and satisfying $\pi_i \circ \pi_i = \pi_i$. The algebraic projectors π_i are called as the algebraic Künneth projectors.*

Definition 2.2. *The motive of X is furthermore said to have a Chow–Künneth decomposition if the algebraic Künneth projectors are orthogonal projectors, i.e., $\pi_i \circ \pi_j = \delta_{i,j}\pi_i$ and $\Delta_X = \oplus_{i=0}^{2n}\pi_i$ in the rational Chow ring of $X \times X$.*

3. Rational homogeneous bundles over a variety

In this section, we firstly recall the motive of a rational homogeneous variety and later construct relative Chow–Künneth projectors for a bundle of homogeneous varieties. The criterion of [GHM2] can then be applied to obtain absolute Chow–Künneth projectors on the total space of the bundle.

3.1. **The motive of a rational homogeneous space.** Let G be a reductive linear algebraic group and P be a parabolic subgroup of G. Then $F := G/P$ is a complete variety. Notice that F is a cellular variety, i.e., it has a cellular decomposition

$$\emptyset = F_{-1} \subset F_0 \subset ... \subset F_n = F$$

such that each $F_i \subset F$ is a closed subvariety and $F_i - F_{i-1}$ is an affine space.

Then we have

Lemma 3.1. [Ko, Theorem, p.363] *The Chow motive $h(F) = (F, \Delta_F)$ of F decomposes as a direct sum of twisted Tate motives*

$$h(F) = \bigoplus_{\omega} \mathbb{L}^{\otimes dim\ \omega}.$$

Here ω runs over the set of cells of F.

3.2. **The motive of a rational homogeneous bundle.** Consider a rational homogeneous bundle $f : Z \longrightarrow S$, i.e., π is a smooth projective morphism and any fibre $\pi^{-1}y$ is a rational homogeneous variety G/P, for some reductive linear algebraic group G and a parabolic subgroup P of G. Assume that S is a smooth complex projective variety.

Consider a rational homogeneous bundle $Z \to S$ defined over $\mathbb{C}$.

Then we consider an étale cover $U \to S$ such that the pullback bundle

$$Z_U := Z \times_S U \to U$$

has a rational section, i.e., a section defined over an open set.

By a theorem due to Colliot-Thélène-Ojanguren [Co-Oj], the bundle $Z_U \to U$ is a Zariski locally trivial fibration.

We want to obtain relative Chow–Künneth projectors for the bundle Z/S. For this purpose, we first construct relative projectors over the étale site of $Z \to S$ and project them to $Z \to S$. We do this by recalling few facts on the rational Chow groups of stacks, which we will essentially apply to the simplest situation—the rational homogeneous bundle $Z \to S$.

3.3. **Chow groups of an étale site.** Mumford, Gillet ([Mu],[Gi]) have defined Chow groups for Deligne–Mumford stacks. Let $\mathcal{X}$ be a smooth DM-stack. The coarse moduli space of $\mathcal{X}$ is denoted by X and $p : \mathcal{X} \to X$ be the projection. So from [Gi, Theorem 6.8], the pullback p^* and pushforward maps p_* establish a ring isomorphism of rational Chow groups

$$CH^*(\mathcal{X})_{\mathbb{Q}} \cong CH^*(X)_{\mathbb{Q}}. \tag{1}$$

This can be applied to the product $p \times p : \mathcal{X} \times \mathcal{X} \to X \times X$, to get a ring isomorphism

$$CH^*(\mathcal{X} \times \mathcal{X})_{\mathbb{Q}} \cong CH^*(X \times X)_{\mathbb{Q}}. \tag{2}$$

Assume that X is a smooth projective variety. Then these isomorphisms also hold in the rational singular cohomology of $\mathcal{X}$ and $\mathcal{X} \times \mathcal{X}$ (for example see [Be]):

$$H^*(\mathcal{X})_{\mathbb{Q}} \cong H^*(X)_{\mathbb{Q}}. \tag{3}$$

and

$$H^*(\mathcal{X} \times \mathcal{X})_{\mathbb{Q}} \cong H^*(X \times X)_{\mathbb{Q}}. \tag{4}$$

Via these isomorphisms, we can pullback the Künneth decomposition of $H^{2n}(X \times X, \mathbb{Q})$ to a decomposition of $H^{2n}(\mathcal{X} \times \mathcal{X}, \mathbb{Q})$, whose components we refer to as the Künneth components of $\mathcal{X}$.

Suppose $Z \to S$ is a rational homogeneous bundle over a smooth projective variety S. Let S^{et} be the étale site on S, together with the natural morphism of the sites $f : S^{et} \to S$.

Here S is considered with the Zariski site. Consider the pullback bundle

$$Z^{et} := Z \times_S S^{et} \to S^{et}$$

over S^{et}.

Given any étale open cover $U \to S$, consider the pullback bundle $Z_U \to U$. The rational relative Chow group $CH^*(Z_U/U)_{\mathbb{Q}}$ of Z_U/U is defined as follows:

$$CH^*(Z_U/U)_{\mathbb{Q}} := CH^*(Z_U)/CH^*(U).$$

Similarly, we define the rational relative Chow groups of Z^{et}/S^{et} as

$$CH^*(Z^{et}/S^{et})_{\mathbb{Q}} := CH^*(Z^{et})_{\mathbb{Q}}/CH^*(S^{et})_{\mathbb{Q}}.$$

Notice that these relative groups can also be defined for other cohomology theories, in particular, for the singular cohomology theory which we denote by $H^*(Z/S)_{\mathbb{Q}}$ and $H^*(Z^{et}/S^{et})_{\mathbb{Q}}$.

Since we are dealing with a rational homogeneous bundle, we can describe these groups explicitly as follows; suppose $\sqcup_{\alpha\in I}\{U_\alpha\} \to S$ is an étale surjective covering of S, for an indexing set I. Moreover, by [Co-Oj], we can assume that the pullback bundles $Z_{U_\alpha} \to U_\alpha$, for $\alpha \in I$, are Zariski locally trivial. Hence $Z_{U_\alpha} \to U_\alpha$ is a relative cellular variety for each $\alpha \in I$. The description of the rational Chow groups of relative cellular spaces $\pi : X \to T$ is given by B. Koeck [Ko] (see also [Ne-Za, Theorem 5.9]), which is stated for the higher Chow groups:

Suppose $X \to T$ is a relative cellular space.

Then there is a sequence of closed embeddings

$$\emptyset = Z_{-1} \subset Z_0 \subset ... \subset Z_n = X$$

such that $\pi_k : Z_k \longrightarrow T$ is a flat projective T-scheme. Furthermore, for any $k = 0, 1, ..., n$, the open complement $Z_k - Z_{k-1}$ is T-isomorphic to an affine space $\mathbb{A}^{m_k}_T$ of relative dimension m_k. Denote $i_k : Z_k \hookrightarrow X$.

Theorem 3.2. *For any $a, b \in \mathbb{Z}$, the map*

$$\begin{array}{ccc} \bigoplus_{k=0}^{n} H_{a-2m_k}(T, b-m_k) & \longrightarrow & H_a(X, b) \\ (\alpha_0, ..., \alpha_n) & \mapsto & \sum_{k=0}^{n} (i_k)_* \pi_k^* \alpha_k \end{array}$$

is an isomorphism. Here $H_a(T, b) = CH_b(T, a-2b)$ are the higher Chow groups of T.

Proof. See [Ko, Theorem, p.371]. □

The above theorem can equivalently be restated to express the rational Chow groups of X as

$$CH^r(X)_{\mathbb{Q}} = \bigoplus_{k=0}^{r} (\oplus_\gamma \mathbb{Q}[\omega_k^\gamma]).f^*CH^k(T)_{\mathbb{Q}}, \tag{5}$$

Here ω_k^γ are the $r-k$ codimensional relative cells and γ runs over the indexing set of $r-k$ codimensional relative cells in the T-scheme X.

Going back to the relative cellular variety $f_\alpha : Z_{U_\alpha} \to U_\alpha$, the above result of Koeck gives a splitting of the quotient map

$$CH^*(Z_{U_\alpha})_{\mathbb{Q}} \to CH^*(Z_{U_\alpha})_{\mathbb{Q}}/CH^*(U_\alpha)_{\mathbb{Q}} =: CH^*(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}}.$$

This gives a natural isomorphism

$$CH^r(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}} = \bigoplus_{k=0}^{r-1} (\oplus_\gamma \mathbb{Q}[\omega_k^\gamma]).f_\alpha^*CH^k(U_\alpha)_{\mathbb{Q}}.$$

For our applications, it suffices to consider the piece $k = 0$, which consists of only the relative algebraic cells of codimension r, namely,

$$RCH^r(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}} := \oplus_\gamma \mathbb{Q}[\omega_0^\gamma].$$

Similarly, the result of [Ko], [Ne-Za] holds in the rational singular cohomology of $Z_{U_\alpha} \to U_\alpha$. So we can also define the piece

$$RH^{2r}(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}} := \oplus_\gamma \mathbb{Q}[\omega_0^\gamma]$$

in the rational singular cohomology of $Z_{U_\alpha} \to U_\alpha$ and the piece

$$RH^{2r}(Z/S)_{\mathbb{Q}} := \oplus_\gamma \mathbb{Q}[\omega_0^\gamma]$$

as a subspace of the rational Betti cohomology $H^{2r}(Z)_{\mathbb{Q}}$, generated by the relative analytic cells ω_0^γ.

Lemma 3.3. *The cycles ω_0^γ in $RCH^*(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}}$ patch together in the étale site to determine a subspace $RCH^*(Z^{et}/S^{et})_{\mathbb{Q}}$ of $CH^*(Z^{et})_{\mathbb{Q}}$, generated by the patched cycles and which maps isomorphically onto the subspace $RH^{2r}(Z/S)_{\mathbb{Q}} \subset H^{2r}(Z)_{\mathbb{Q}}$, under the cycle class map*

$$CH^*(Z^{et})_{\mathbb{Q}} \to H^{2*}(Z^{et})_{\mathbb{Q}} \simeq H^{2*}(Z)_{\mathbb{Q}}.$$

Proof. Using the isomorphism in (3), there is an isomorphism

$$H^{2*}(Z^{et})_{\mathbb{Q}} \simeq H^{2*}(Z)_{\mathbb{Q}}.$$

Hence the cycles $\omega_0^\gamma \in RCH^*(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}}$ patch together as analytic cycles in the étale site and determine a subspace $RH^{2r}(Z^{et}/S^{et})_{\mathbb{Q}} \subset H^{2r}(Z^{et})_{\mathbb{Q}}$, mapping isomorphically onto $RH^{2r}(Z/S)_{\mathbb{Q}} \subset H^{2r}(Z)_{\mathbb{Q}}$.

Now, by definition, there is a natural isomorphism

$$RCH^*(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}} \xrightarrow{\simeq} RH^{2*}(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}} \tag{6}$$

between the 0-th piece of the rational relative Chow groups and the relative Betti cohomology, for each α.

Via the isomorphism in (6), the patching conditions required over the étale site, to define the piece $RCH^{2r}(Z^{et}/S^{et})_{\mathbb{Q}}$ are the same as those for $RH^{2r}(Z^{et}/S^{et})_{\mathbb{Q}}$. More precisely, the patching conditions are given in [Mu], [Gi, §4] and the identification in (6) together with the fact that the patching conditions are fulfilled for the singular cohomology of the étale site, i.e., the cycles ω_0^{γ} patch together to give a class in $RH^{2r}(Z^{et}/S^{et})_{\mathbb{Q}}$, and hence they also patch together to give a class in $RCH^{2r}(Z^{et}/S^{et})_{\mathbb{Q}}$. These patched classes generate the $\mathbb{Q}$-subspace $RCH^{2r}(Z^{et}/S^{et})_{\mathbb{Q}} \subset CH^*(Z^{et})_{\mathbb{Q}}$ and which maps isomorphically onto the subspace $RH^{2r}(Z/S)_{\mathbb{Q}} \subset H^{2r}(Z)_{\mathbb{Q}}$ under the cycle class map. □

Corollary 3.4. *There is a canonical isormorphism*

$$RCH^r(Z^{et}/S^{et})_{\mathbb{Q}} \simeq RH^{2r}(Z^{et}/S^{et})_{\mathbb{Q}}.$$

Let $n := \dim(Z/S)$.

Corollary 3.5. *The bundle $Z \to S$ has a relative Chow–Künneth decomposition, in the sense of* [GHM].

Proof. This is a corollary of Lemma 3.3. We apply the ring isomorphisms of [Be] to the relative groups of the product spaces $(Z^{et} \times_{S^{et}} Z^{et}) \to S^{et}$, $(Z \times_S Z) \to S$ and notice that the relative orthogonal Künneth projectors in $H^{2n}(Z \times_S Z/S, \mathbb{Q})$ lift to orthogonal projectors in $H^{2n}(Z^{et} \times_{S^{et}} Z^{et}/S^{et}, \mathbb{Q})$ and which add to the relative diagonal cycle. Now we note that the relative diagonal $\Delta_{Z/S}$ and its orthogonal Künneth components actually lie in the piece $RH^{2n}(Z \times_S Z/S)_{\mathbb{Q}}$ (generated by the relative algebraic cells) and under the above isomorphism of Behrend, lift to an orthogonal decomposition

$$\Delta_{Z^{et}/S^{et}} = \sum_{i=0}^{2n} \Pi_i \in RH^{2n}(Z^{et} \times_{S^{et}} Z^{et}/S^{et})_{\mathbb{Q}}$$

over the étale site. By Corollary 3.4 applied to the product space $Z^{et} \times_{S^{et}} Z^{et} \to S^{et}$, the above orthogonal projectors correspond to orthogonal algebraic projectors in $RCH^n(Z^{et} \times_{S^{et}} Z^{et}/S^{et})_{\mathbb{Q}}$, which add to the relative diagonal cycle $\Delta_{Z^{et}/S^{et}}$. Now, the ring isomorphism in (2) restricted to the rational relative Chow groups of the étale site and the rational relative Chow groups of the Zariski site says that the relative orthogonal algebraic projectors can be descended to the rational relative Chow groups of $(Z \times_S Z)/S$, to give relative Chow–Künneth projectors and a relative Chow–Künneth decomposition

$$\Delta_{Z/S} = \sum_{i=0}^{2n} \Pi_i \in CH^n(Z \times_S Z)_{\mathbb{Q}}.$$

□

Proposition 3.6. *Suppose $Z \to S$ is a rational homogeneous bundle over a smooth variety S. Then the motive of the bundle $Z \to S$ is expressed as a sum of tensor products* *of summands of the motive of S with the twisted Tate motive. More precisely, the motive of Z can be written as*

$$h(Z) = \bigoplus_i h^i(Z)$$

where $h^i(Z) = \bigoplus_{j+k} r_{\omega_\alpha}.\mathbb{L}^j \otimes h^k(S)$. Here r_{ω_α} is the number of j-codimensional cells on a fibre $\mathbb{F}$

In particular, if S has a Chow–Künneth decomposition then Z also admits an absolute Chow–Künneth decomposition.

Proof. By Corollary 3.5, we know that the bundle Z/S has a relative Chow–Künneth decomposition. Since the map $Z \to S$ is a smooth morphism and the fibres of $Z \to S$ have only algebraic cohomology, we can directly apply the criterion in [GHM2, Main theorem 1.3], to get absolute Chow–Künneth projectors for Z and the decomposition stated above (for example, see [Iy, Lemma 3.2, Corollary 3.3]). □

Remark 3.7. *Suppose X is a smooth projective variety with a nef tangent bundle. Then by* [Ca-Pe],[DPS]*, we know that there is an étale cover $X' \to X$ of X such that $X' \to A$ is a smooth morphism over an abelian variety A, whose fibres are smooth Fano varieties with a nef tangent bundle. It is an open question* [Ca-Pe, p.170]*, whether such a Fano variety is a rational homogeneous variety. A positive answer to this question, together with Proposition 3.6, will give absolute Chow–Künneth projectors for any variety with a nef tangent bundle.*

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