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Projective normality of abelian surfaces given by primitive line bundles

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Abstract. In this paper, we study projective normality of abelian surfaces, with embeddings given by ample line bundles of type $(1, d)$. We show that if $d \geq 7$, the generic abelian surface is projectively normal.

1. Introduction

Let L be an ample line bundle of type $\delta = (\delta_1, \delta_2, \ldots, \delta_g)$ on an abelian variety A of dimension g. It is known that the embedding of A given by the morphism ϕ_{L^n} associated to L^n is projectively normal if $n \geq 3$ and when $n = 2$, no point of $K(L^2)$ is a base point for L, (see [2], 7.3.1).

Here we consider the case of an abelian surface when L is primitive i.e. δ is of the form $(1, d)$. When $d \leq 4$, L is not even very ample, while for $d = 5, 6$, dimension of $\mathrm{Sym}^2 H^0(L)$ is less than that of $H^0(L^2)$ so that L can never be projectively normal. Here we show

Theorem 1.1. *Let L be an ample line bundle of type $D = (1, d)$ on an abelian surface A. Suppose the Neron Severi group of A is isomorphic to $\mathbb{Z}$ and $d \geq 7$, then the image of A, under the morphism $\phi_L : A \longrightarrow \mathbb{P} = P(H^0(L)^\vee)$ associated to L, is projectively normal.*

This is the generic situation, by [7]. Lazarsfeld has shown (see [4]) that this is true for $d \geq 13$ and for $d = 7, 9, 11$ whenever L is very ample. Our method is different even in these cases. We show that if L is not projectively normal, then the corresponding imbedding into the complete linear system, takes A into a quadric of rank less than or equal to 4, which can be ruled out if *Neron Severi* number is 1. This results from an analysis of the representations of the theta group $\mathcal{G}(L)$ on $\mathrm{Sym}^2 H^0(L)$ as well as on $H^0(L^2)$.

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2. Preliminaries

Let L be an ample line bundle of type $\delta = (d_1, d_2, \ldots, d_g)$ on an abelian variety A of dimension g.

Suppose L is a symmetric line bundle i.e. $L \simeq i^*L$, where i is the involution $a \longrightarrow -a$ on A. The action of i can be lifted to the line bundle L, and we may normalise this action by requiring that this action still be an involution and that its action on the fibre of L at 0 be the identity. Consider the function e_*^L which associates to any $a \in A_2$, the element $+1$ or -1, according as the action of i on the fibre L_a of L at a, is $+1$ or -1.

Definition 2.1. *A symmetric line bundle L on an abelian variety A is said to be strongly symmetric, (see [1]), if $e_*^L(a) = +1$, for all elements in $A_2 \cap K(L)$, where $K(L) = \{a \in A : L \simeq t_a^* L\}$ (t_a being the translation map by a on A).*

Consider the natural Heisenberg extension $\mathcal{G}(L)$ of $K(L)$ by $\mathbb{C}^*$, and the natural Heisenberg extension Heis (δ) of $K(\delta) = (\mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_g\mathbb{Z})^2$ by $\mathbb{C}^*$.

Recall the following homomorphisms, defined as in [5], if L is symmetric,

$$\begin{aligned}
\epsilon_2 &: \mathcal{G}(L) \longrightarrow \mathcal{G}(L^2)\\
\eta_2 &: \mathcal{G}(L^2) \longrightarrow \mathcal{G}(L)\\
\delta_n &: \mathcal{G}(L) \longrightarrow \mathcal{G}(L)\\
E_2 &: \mathrm{Heis}\,(\delta) \longrightarrow \mathrm{Heis}\,(2\delta)\\
D_n &: \mathrm{Heis}\,(\delta) \longrightarrow \mathrm{Heis}\,(\delta).
\end{aligned}$$

Then one has,

$$\begin{aligned}
\delta_2 &= \eta_2 \circ \epsilon_2 \ \text{ for } \ \mathcal{G}(L)\\
\delta_2 &= \epsilon_2 \circ \eta_2 \ \text{ for } \ \mathcal{G}(L^2) \quad \text{(See [5])}.
\end{aligned}$$

Definition 2.2. *For a line bundle L of type δ, an isomorphism $f : \mathcal{G}(L) \longrightarrow$ Heis (δ), which restricts to identity on $\mathbb{C}^*$, is called a theta structure for L. Moreover, suppose L is symmetric, a theta structure $f : \mathcal{G}(L) \longrightarrow \mathcal{G}(\delta)$ is called symmetric if $f \circ \delta_{-1} = D_{-1} \circ f$.*

Proposition 2.3. *Let L be a symmetric line bundle of type δ on an abelian variety A of dimension g. Then L is strongly symmetric if and only if it admits symmetric theta structures.*

Proof. Suppose L is a strongly symmetric line bundle on A. Consider an isomorphism, say h, of $K(L)$ onto $K(\delta)$, which respects the corresponding Weil pairing of the Heisenberg extensions. By [1],15.6, h is induced by a symmetric theta structure $f : \mathcal{G}(L) \longrightarrow$ Heis (δ).

Conversely, if L admits a symmetric theta structure, say f, then by definition, $f \circ \delta_{-1} = D_{-1} \circ f$. Consider an element of order 2, say w, in $K(L)$. Let z be a lift of w in $\mathcal{G}(L)$. By [5], Proposition 3, p. 309,

$$\delta_{-1}(z) = e_*^L(w).z$$

On the other hand, by definition of D_{-1}, $D_{-1}(f(z)) = f(z)$. Hence

$$\begin{aligned} f \circ \delta_{-1}(z) &= f(e_*^L(w).z) \\ &= e_*^L(w).f(z) \\ &= e_*^L(w).D_{-1}(f(z)). \end{aligned}$$

implying $e_*^L(w) = +1$. □

Remark 2.4. By [2], 6.9.5, if H is a polarization of type $\delta = (d_1, d_2, \ldots, d_g)$, with $d_1, \ldots, d_s$ odd and $d_{s+1}, \ldots, d_g$ even, then there are 2^{2s} symmetric line bundles in $Pic^H(X)$ admitting symmetric theta structures. Hence by 2.1, there are 2^{2s} strongly symmetric line bundles in $Pic^H(X)$.

Proposition 2.5. *For a pair (A, L), as in 1.1, with L strongly symmetric, there exist symmetric theta structures f_1 and f_2 such that the following diagram*

$$\begin{array}{ccc} \mathcal{G}(L) & \xrightarrow{f_1} & \text{Heis}\,(\delta) \\ \downarrow \epsilon_2 & & \downarrow E_2 \\ \mathcal{G}(L^2) & \xrightarrow{f_2} & \text{Heis}\,(2\delta) \end{array}$$

commutes.

Proof. Consider the homomorphism $\eta_{(x,l)} : \text{Heis}\,(2\delta) \longrightarrow \text{Heis}\,(\delta)$, where (x, l) is an order 2 element in $K(2\delta)$, given as $\eta_{(x,l)}(\alpha, y, m) = (\alpha^2 l(y) m(x), 2y, m')$, where m' is the image in $K_1(\delta)^\vee$ of m induced by the inclusion $K_1(\delta) \xrightarrow{\times 2} K_1(2\delta)$.

By [1], 16.19, there exist (x, l), an element of order 2 in $H(2\delta)$, and symmetric theta structures f_1 and f_2, such that the following diagram,

$$\begin{array}{ccc} \mathcal{G}(L^2) & \xrightarrow{f_2} & \text{Heis}\,(2\delta) \\ \downarrow \eta_2 & & \downarrow \eta_{(x,l)} \\ \mathcal{G}(L) & \xrightarrow{f_1} & \text{Heis}\,(\delta) \end{array}$$

commutes. Observe that $E_2 \circ \eta_{(x,l)} = D_2$ for Heis (2δ). Now consider,

$$\begin{aligned}
f_2 \circ \epsilon_2(\eta_2(z)) &= f_2(\delta_2(z)) \\
&= f_2((z^3).\delta_{-1}(z)) \\
&= f_2(z^3).f_2(\delta_{-1}(z)) \\
&= f_2(z)^3.D_{-1}(f_2(z)) \\
&= D_2(f_2(z)) \\
&= E_2 \circ \eta_{(x,l)}(f_2(z)) \\
&= E_2 \circ f_1(\eta_2(z))
\end{aligned}$$

hence $f_2 \circ \epsilon_2 = E_2 \circ f_1$. $\square$

3. Representations of the Heisenberg group with central charge 2

An irreducible representation of the theta group $\mathcal{G}(L)$ such that $\alpha \in \mathbb{C}^*$ acts as $\alpha \mapsto \alpha^n$ is said to be of *central charge n*.

Proposition 3.1. *Let L be an ample line bundle of type (d_1, d_2) on an abelian surface A. Then any irreducible representation of the theta group $\mathcal{G}(L)$, with central charge n, is of dimension at least $\frac{d_1 d_2}{(n,d_1).(n,d_2)}$, where (n, d_i) denotes the greatest common divisor of n and d_i. In particular, if the type is $(1, d)$ and $n = 2$, then every irreducible representation is of dimension at least d, if d is odd, and at least $d/2$, if d is even.*

Proof. See [2], Ex. 6.4.a. $\square$

Now, we prove some statements on irreducible representations of Heis (δ) with central charge 2, and these strengthen 3.1.
Consider the standard Heisenberg group Heis (δ), of type $\delta = (\delta_1, \delta_2, \ldots, \delta_g)$. Then we have the short exact sequence

$$(1)\quad 1 \longrightarrow \mathbb{C}^* \longrightarrow \text{Heis}\,(\delta) \longrightarrow K(\delta) \longrightarrow 0$$

where $K(\delta) = (\mathbb{Z}/\delta\mathbb{Z})^2$.

Proposition 3.2. *The set of isomorphism classes of irreducible representations of* Heis (δ) *with central charge 2 is in bijection with the set of quadratic forms on $K(\delta)_2$ whose associated bilinear form is the restriction of the Weil form e^δ to $K(\delta)_2$. Moreover, each such representation has dimension equal to $|\delta|/\sqrt{|K(\delta)_2|}$, where $|\delta| = \delta_1 \ldots \delta_g$.*

Proof. Let ρ be an irreducible representation of Heis (δ) with central charge 2. Then, ρ descends to a representation of Heis $(\delta)/\{\pm 1\}$ with central charge 1. Call it ρ again. We have a diagram with exact rows,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \text{Heis}\,(\delta) & \xrightarrow{\eta} & K(\delta) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \text{Heis}\,(\delta)/\{\pm 1\} & \xrightarrow{\psi} & K(\delta) & \longrightarrow & 0 \end{array}$$

The left vertical map is the homomorphism $z \mapsto z^2$. Notice that $K(\delta)_2$ is isotropic for $(e^\delta)^2$, since if $x, y \in K(\delta)_2$, $(e^\delta)^2(x, y) = e^\delta(2x, y) = 1$. In fact, it is the *nilradical* of $(e^\delta)^2$. In particular, if s is a splitting over $K(\delta)_2$ in the second row, then

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \text{Heis}\,(\delta)/\{\pm 1, s(K(\delta)_2\} \longrightarrow K(\delta)/K(\delta)_2 \longrightarrow 0$$

is a standard Heisenberg extension.

Also the subgroup $F = \psi^{-1}(K(\delta)_2)$ is the centre of Heis $(\delta)/\{\pm 1\}$. Hence in the representation ρ, the subgroup F acts as a character, say χ_ρ, such that $\chi_\rho(\alpha) = \alpha$ for $\alpha \in \mathbb{C}^*$. We call such a character, a character with central charge 1. We first show that this correspondence is a bijection. Let ρ_1 and ρ_2 be two irreducible representations of Heis $(\delta)/\{\pm 1\}$, with central charge 1. Suppose $\chi_{\rho_1} = \chi_{\rho_2}$. Since we have chosen the splitting s, χ_{ρ_1} corresponds to a character on $K(\delta)_2$ which extends to a character on $K(\delta)$. Call it χ'. Then $\chi = \chi' \circ \psi$ is a character on Heis $(\delta)/\{\pm 1\}$ which is trivial on $\mathbb{C}^*$. Notice that $s(K(\delta)_2)$ acts trivially in $\rho_1 \otimes \chi^{-1}$ as well as in $\rho_2 \otimes \chi^{-1}$. Hence these representations descend to irreducible representations of $G = \text{Heis}\,(\delta)/\{\pm 1, s(K(\delta)_2)\}$ with central charge 1. Since G has a unique irreducible representation, upto scalars, with central charge 1, we conclude that $\rho_1 \otimes \chi^{-1} \simeq \rho_2 \otimes \chi^{-1}$ as a G-module. This gives $\rho_1 \simeq \rho_2$ as a Heis $(\delta)/\{\pm 1\}$-module. Conversely, given a character on F, it corresponds to a character χ on Heis $(\delta)/\{\pm 1\}$ of central charge 1, as shown above. Hence if σ is the unique irreducible representation of G and p : Heis $(\delta)/\{\pm 1\} \longrightarrow G$ is the quotient map, then $(\sigma \circ p) \otimes \chi$ gives an irreducible representation of Heis $(\delta)/\{\pm 1\}$ with central charge 1. Clearly dimension of any such representation is $\sqrt{|K(\delta)/K(\delta)_2|} = |\delta|/\sqrt{|K(\delta)_2|}$. Now a character χ' on F gives a character χ on $\eta^{-1}(K(\delta)_2)$ such that $\chi(\alpha) = \alpha^2$ for all $\alpha \in \mathbb{C}^*$. Define a quadratic form $q : \eta^{-1}(K(\delta)_2) \longrightarrow \mathbb{C}^*$ as follows, $x \mapsto x^2\chi^{-1}(x)$. One easily sees that $q(\lambda x) = q(x)$, for $\lambda \in \mathbb{C}^*$ and $q(x.y) = e^\delta(x, y)q(x)q(y)$. Hence q corresponds to a quadratic form on $K(\delta)_2$ whose associated bilinear form is the restriction of the Weil form e^δ to $K(\delta)_2$. □

Let $V(\delta)$ be the unique irreducible representation of Heis (δ) with central charge 1 and let Q be the set of all quadratic forms on $K(\delta)_2$ whose associated bilinear form is the restriction of e^δ to $K(\delta)_2$.

Lemma 3.3. *The vector space $V(\delta) \otimes V(\delta)$ is a* Heis (δ)*-module with central charge 2 and splits into isotypical components corresponding to $q \in Q$, each of dimension equal to $|\delta|^2/|K(\delta)_2|$.*

Proof. Let $G = (\text{Heis}(\delta) \times \text{Heis}(\delta))/\{(\lambda, \lambda^{-1}) : \lambda \in \mathbb{C}^*\}$. Then by [6], 1.1.5,

$$1 \longrightarrow \mathbb{C}^* \longrightarrow G \longrightarrow K(\delta) \times K(\delta) \longrightarrow 0$$

is a Heisenberg extension with Weil pairing $e^\delta \times e^\delta$ and the vector space $V(\delta) \otimes V(\delta)$ is the unique irreducible representation of G with central charge 1. Notice that $s(K(\delta)_2) \subset \text{Heis}(\delta)/\{\pm 1\} \overset{\Delta}{\hookrightarrow} G$ and $K(\delta)_2 \overset{\Delta}{\hookrightarrow} K(\delta) \times K(\delta)$ is isotropic for $e^\delta \times e^\delta$. Hence $V(\delta) \otimes V(\delta) = \sum_{\chi \in \widehat{s(K(\delta)_2)}} I_\chi$. Clearly dim $I_\chi = |\delta|^2/|K(\delta)_2|$. Since $\chi \in \widehat{s(K(\delta)_2)}$ correspond to quadratic forms as shown in 3.2, the proposition is proved. $\square$

Now, $\text{Sym}^2(V(\delta))$ and $\wedge^2 V(\delta)$ are Heis $(\delta)/\{\pm 1\}$-modules and we wish to determine the isotypical decomposition of these vector spaces for any type δ. Since $K(\delta)$ can be written as $K(\delta_1) \times K(\delta_2)$ such that $\delta_1 = (2, 2, \ldots 2)$ and $\delta_2 = (\delta_1, \delta_2, \ldots, \delta_g)$, where if δ_i is even then it is divisible by 4, the Heisenberg group $\text{Heis}(\delta) = \text{Heis}(\delta_1) \times \text{Heis}(\delta_2)/\{(\lambda, \lambda^{-1}) : \lambda \in \mathbb{C}^*\}$ and the unique irreducible representation $V(\delta)$ can be written as $V(\delta_1) \otimes V(\delta_2)$. Hence it is enough to study the cases when $\delta = \delta_1$ and $\delta = \delta_2$.

Remark 3.4. When $\delta = (2, 2, \ldots, 2)$ it is well known that $\text{Sym}^2 V(\delta) = \sum_{q \in Q,\, Arf(q)=0} E_q$ and $\wedge^2 V(\delta) = \sum_{q \in Q,\, Arf(q)=1} E_q$, where $Arf(q)$ denotes the Arf invariant of q and E_q are the isotypical components corresponding to q of dimension 1.

Proposition 3.5. *Suppose $\delta = (\delta_1, \delta_2, \ldots, \delta_g)$, such that if δ_i is even then it is divisible by 4. Then the isotypical decomposition is given as follows,*

$$\begin{aligned}\text{Sym}^2 V(\delta) = &\sum_{\chi \in \widehat{K(\delta)_2}, \chi \neq 1} |\delta|/(2\sqrt{|K(\delta)_2|}).\rho_\chi \\ &+ (|\delta| + |K(\delta)_2|)/(2\sqrt{|K(\delta)_2|}).\rho_1\end{aligned}$$

and

$$\begin{aligned}\wedge^2 V(\delta) = &\sum_{\chi \in \widehat{K(\delta)_2}, \chi \neq 1} |\delta|/(2\sqrt{|K(\delta)_2|}).\rho_\chi \\ &+ (|\delta| - |K(\delta)_2|)/(2\sqrt{|K(\delta)_2|}).\rho_1,\end{aligned}$$

where ρ_χ is the irreducible representation of Heis (δ) *with central charge 2, corresponding to the character χ on $K(\delta)_2$.*

Proof. Notice that $K(\delta)_2 \subset 2K(\delta)$. Hence $K(\delta)_2$ is isotropic for e^δ, since if $x, y \in K(\delta)_2$, $x = 2x'$ and $e^\delta(x, y) = e^\delta(x', y)^2 = e^\delta(x', 2y) = 1$. Thus the quadratic forms on $K(\delta)_2$ are actually characters on $K(\delta)_2$. Hence if s is a splitting over $K(\delta)_2$ in (1) then $V(\delta) = \sum_{\chi \in \widehat{K(\delta)_2}} W_\chi$, where $W_\chi = \{f \in V(\delta) : s(x)f = \chi(x)f \ for\ x \in K(\delta)_2\}$ and $\dim W_\chi = |\delta|/|K(\delta)_2|$. Hence $V(\delta) \otimes V(\delta) = \sum_{\chi \in \widehat{K(\delta)_2}} I_\chi$, where $I_\chi = \sum_{\chi_1 . \chi_2 = \chi} W_{\chi_1} \otimes W_{\chi_2}$ is the isotypical decomposition as Heis $(\delta)/\{\pm 1\}$-module.

Now consider the involution j on $V(\delta) \otimes V(\delta)$ given as $j(x \otimes y) = y \otimes x$. Then

$$\mathrm{Sym}^2 V(\delta) = \{f \in V(\delta) \otimes V(\delta) : jf = f\}$$

and

$$\wedge^2 V(\delta) = \{f \in V(\delta) \otimes V(\delta) : jf = -f\}.$$

Obviously, if $f \in W_{\chi_1} \otimes W_{\chi_2}$ then $jf \in W_{\chi_2} \otimes W_{\chi_1}$. Let $\mathrm{Sym}^2 V(\delta) = \sum_{\chi \in \widehat{K(\delta)_2}} S_\chi$ and $\wedge^2 V(\delta) = \sum_{\chi \in K(\delta)_2^\vee} \wedge_\chi$ be the isotypical decomposition as a Heis (δ)-module with central charge 2, where S_χ and $\wedge_\chi$ are the isotypical components. In fact,

$$(2) \quad S_\chi = \mathbb{C}\{f + jf : f \in W_{\chi_1} \otimes W_{\chi_2} \text{ and } \chi_1 . \chi_2 = \chi\}$$

and

$$(3) \quad \wedge_\chi = \mathbb{C}\{f - jf : f \in W_{\chi_1} \otimes W_{\chi_2} \text{ and } \chi_1 . \chi_2 = \chi\}.$$

Since $I_\chi = |\delta|/\sqrt{|K(\delta)_2|}.\rho_\chi$ and $I_\chi = S_\chi + \wedge_\chi$, it follows that when χ is nontrivial, $S_\chi = |\delta|/\sqrt{|K(\delta)_2|}.\rho_\chi$ and $\wedge_\chi = |\delta|/\sqrt{|K(\delta)_2|}.\rho_\chi$. When χ is trivial, $S_1 = \sum_{\chi \in K(\delta)_2^\vee} \mathrm{Sym}^2 W_\chi$. Since $\dim W_\chi = |\delta|/|K(\delta)_2|$, $\dim \mathrm{Sym}^2 W_\chi = |\delta|(|\delta| + |K(\delta)_2|/(2|K(\delta)_2|^2)$, we conclude that

$$S_1 = (|\delta| + |K(\delta)_2|/(2\sqrt{|K(\delta)_2|}).\rho_1$$

and

$$\wedge_1 = (|\delta| - |K(\delta)_2|)/(2\sqrt{|K(\delta)_2|}).\rho_1 \qquad \square$$

Proposition 3.6. *Let* $\mathrm{Sym}^2 V(\delta) = \sum_{q \in Q} m_q \rho_q$ *and* $\wedge^2 V(\delta) = \sum_{q \in Q} n_q \rho_q$ *be the isotypical decomposition of the symmetric and the exterior power of the unique irreducible representation of* Heis (δ) *of central charge 1. Let* $K^\perp$ *be the nilradical of the restriction of* e^δ *to* $K(\delta)_2$*. Then the restriction of any* q *in* Q *to* $K^\perp$ *is a character and if it is trivial, then it induces a quadratic form* q' *on* $K(\delta)_2/K^\perp$*.*

a) If $q|_{K^\perp}$ *is nontrivial then* $m_q = n_q = |\delta|/(2\sqrt{|K(\delta)_2|})$.

b) If $q|_{K^\perp}$ is trivial, then

$$\begin{aligned} m_q &= (|\delta| + \sqrt{|K(\delta)_2|}).\sqrt{|K^\perp|}/(2\sqrt{|K(\delta)_2|}), \ \textit{if } Arf(q') = 0 \\ &= (|\delta| - \sqrt{|K(\delta)_2|}.\sqrt{|K^\perp|})/(2\sqrt{|K(\delta)_2|}), \ \textit{if } Arf(q') = 1. \\ n_q &= (|\delta| - \sqrt{|K(\delta)_2|}).\sqrt{|K^\perp|}/(2\sqrt{|K(\delta)_2|}), \ \textit{if } Arf(q') = 0 \\ &= (|\delta| + \sqrt{|K(\delta)_2|}.\sqrt{|K^\perp|})/(2\sqrt{|K(\delta)_2|}), \ \textit{if } Arf(q') = 1. \end{aligned}$$

Here ρ_q denotes the irreducible representation of Heis $(\delta)/\{\pm 1\}$ *corresponding to $q \in Q$.*

Proof. Clearly q restricted to $K^\perp$ is a character and if $x \in K(\delta)_2, y \in K^\perp$ then $q(x+y) = q(x)q(y)e^\delta(x,y) = q(x)$. Hence q descends to a quadratic form q' on $K(\delta)_2/K^\perp$. Recall that one can write $K(\delta) = K(\delta_1) \times K(\delta_2)$, Heis $(\delta) =$ (Heis $(\delta_1) \times$ Heis $(\delta_2))/\{(\lambda, \lambda^{-1}) : \lambda \in \mathbb{C}^*\}$ and $V(\delta) = V(\delta_1) \otimes V(\delta_2)$ where δ_1 and δ_2 are as in 3.4 and 3.5 respectively. Then we observe that

1) $K(\delta_1)$ is isomorphic to $K(\delta)_2/K^\perp$ and
2) $K(\delta_2)$ is isomorphic to $K^\perp$.

Since $\mathrm{Sym}^2 V(\delta) = \mathrm{Sym}^2 V(\delta_1) \otimes \mathrm{Sym}^2 V(\delta_2) \oplus \wedge^2 V(\delta_1) \otimes \wedge^2 V(\delta_2)$ and the irreducible representation ρ_q of Heis (δ) with central charge 2 is a tensor product of irreducible representations of Heis (δ_1) and Heis (δ_2) of central charge 2, the proposition follows from 3.4 and 3.5. □

Proposition 3.7. *Regard $V(2\delta) = \{f : \mathbb{Z}/2\delta\mathbb{Z} \longrightarrow \mathbb{C}\}$, as a* Heis (δ)-*module via the homomorphism E_2 :* Heis $(\delta) \longrightarrow$ Heis (2δ). *Then its isotypical decomposition is given as follows*

$$V(2\delta) = \sum_{\chi \in \widehat{K(\delta)_2}} (2^g/\sqrt{|K(\delta)_2|}).\rho_\chi.$$

Proof. Recall the homomorphism E_2 : Heis $(\delta) \longrightarrow$ Heis (2δ). Then the irreducible representation of Heis (2δ) with central charge 1, $V(2\delta)$, splits into isotypical components as a E_2(Heis (δ))-module. Since E_2(Heis $(\delta)) =$ Heis $(\delta)/\{\pm 1\}$ is a normal subgroup, each of the isotypical components has the same dimension. Now, given any irreducible representation σ of Heis $(\delta)/\{\pm 1\}$ with central charge 1, consider the induced representation ρ on Heis (2δ). Then ρ is a direct sum of isomorphic copies of $V(2\delta)$, and $\sigma \subset \rho$. Thus every irreducible representation of Heis (δ) with central charge 2, occurs with equal multiplicity in $V(2\delta)$. Since $\dim V(2\delta) = 2^g|\delta|$ and $\dim\sigma = |\delta|/\sqrt{|K(\delta)_2|}$, the proposition is proved. □

4. Proof of Theorem 1.1

We need the following two propositions in our proof of 1.1.

Proposition 4.1. *Let $\mathbb{P}^N$ be the space of all quadrics in $\mathbb{P}^n$ for some fixed $n > 0$. Then the subvariety $\mathcal{V}_k = \{t \in \mathbb{P}^N : q_t$ is a quadric in $\mathbb{P}^n$ of rank $\leq n + 1 - k\}$, of $\mathbb{P}^N$, is of dimension $N - l$, where $l = k(k+1)/2$.*

Proof. See [3], Example 22.31. □

Proposition 4.2. *Let (A, L) be a pair as in* 1.1. *Then there are no quadrics of rank less than or equal to four, containing the image of A, in $\mathbb{P}$.*

Proof. Suppose Q is a quadric of rank less than or equal to four, in $\mathbb{P}$, containing image of A. Clearly rank of Q cannot be ≤ 2 since then A would be contained in a hyperplane. We will denote by Z the inverse image of the nullspace N for the quadric Q. Since the Neron Severi group $NS(A)$ of A is isomorphic to $\mathbb{Z}$, generated by L, the pullback to A of any hyperplane section is an irreducible curve. Also, since codimension of N is at least 4 in $\mathbb{P}$, let H_1 and H_2 be distinct hyperplanes containing N. Then the intersection of pullbacks of H_1 and H_2 to A is a finite set of points.

Case 1: If rank $Q = 4$, then write

$$Q = X_0^2 + X_1^2 + X_2^2 + X_3^2$$

for some basis of $H^0(L)$. Project Q to $T = (X_4 = X_5 = \cdots = X_{d-1} = 0)$. Then one has a finite morphism $A - Z \xrightarrow{p} Q' \subset T$, where Q' is a smooth quadric in T. Now $\mathcal{O}_T(1)$ restricted to Q' which is $\mathbb{P}^1 \times \mathbb{P}^1$ is $p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$, where p_i are projections to $\mathbb{P}^1$, for $i = 1, 2$. Hence $p^*(p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1))$ on $A - Z$ extend to $L_1 \otimes L_2$ on A, where L_1 and L_2 are nontrivial line bundles on A. Hence $L \simeq L_1 \otimes L_2$, which is a contradiction.

Case 2: If rank $Q = 3$, then one has a map $A - Z \longrightarrow C \subset S$, where S is a linear space of dimension 2, defined similarly as in Case 1, and C is a conic. Then $\mathcal{O}_S(1)$ restricts to a divisor of degree 2 on C, hence its pullback on A gives a reducible divisor as in Case 1, which is a contradiction. □

Consider the action of $\mathcal{G}(L)$ on $\mathrm{Sym}^n\, H^0(A, L)$ and $H^0(A, L^n)$. The natural maps

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) = \mathrm{Sym}^n\, H^0(L) \xrightarrow{\rho_n} H^0(A, L^n),$$

are clearly equivariant for the action of $\mathcal{G}(L)$. □

We have to show that ρ_n, for $n \geq 2$, are surjective. Suppose $n = 2$. Then $\dim\, H^0(\mathcal{O}_{\mathbb{P}}(2)) = \frac{d(d+1)}{2}$ and $\dim\, H^0(A, L^2) = 4d$. Hence $\dim \mathrm{Ker}\ \rho_2 \geq \frac{d(d+1)}{2} - 4d$.

Lemma 4.3. *Let $W = \mathrm{Ker}\ \rho_2$. If $dim W > d(d+1)/2 - 4d$, and $d \geq 7$ odd, $d \geq 14$ and d even, there are quadrics of rank less than or equal to 4 in W.*

Proof. Let $R = P(W)$. Then R is a space of quadrics of $\mathbb{P}$ containing image of A. Suppose d is odd. Then dimension of R is at least $d(d+1)/2-3d-1$, by 3.4. Consider the subvariety $\mathcal{V}_k$ defined as in 4.1, for $n = d-1$ and $k = d-4$. It is enough to see that the linear space R intersects the subvariety $\mathcal{V}_k$ in the space of all quadrics of $\mathbb{P}$. This is clear, since

$$\begin{aligned} m - l &= \frac{d(d+1)}{2} - 3d - 1 - \frac{(d-3)(d-4)}{2} \\ &= \frac{2d-14}{2} \geq 0 \ \ if \ \ d \geq 7 \end{aligned}$$

If d is even, then $d = 2r, r \geq 7$. In this case $dim W \geq \frac{d(d+1)}{2} - 4d + r$, by 2.4. One does similar computation as above, to show there are quadrics of rank less than or equal to 4 containing image of A. □

This contradicts 4.2. Hence ρ_2 is surjective if $d \geq 7$, odd and $d \geq 14$.

We now consider the cases when $d = 8, 10,$ and 12.

One may assume L is strongly symmetric, from 2.4. By 2.5, there exist compatible theta structures $b : \mathcal{G}(L) \simeq \mathrm{Heis}\,(\delta)$ and $b^* : \mathcal{G}(L^2) \simeq \mathrm{Heis}\,(2\delta)$. These induce isomorphisms $b' : H^0(L) \longrightarrow V(\delta) = \{f : \mathbb{Z}/d\mathbb{Z} \longrightarrow \mathbb{C}\}$, $b^{*'} : H^0(L^2) \longrightarrow V(2\delta) = \{f : \mathbb{Z}/2 \times \mathbb{Z}/2d\mathbb{Z} \longrightarrow \mathbb{C}\}$, unique upto scalars, and compatible with b and b^*.

Via the isomorphisms $b', b^{*'}$ and 2.5, it is enough to show that the induced Heis (δ)-equivariant map,

$$\rho_2' : \mathrm{Sym}^2(V(\delta)) \longrightarrow V(2\delta)$$

is surjective.

First we consider the case when $\delta = (1, 10)$.

Notice that $\mathrm{Heis}\,(\delta) = (\mathrm{Heis}\,(2) \times \mathrm{Heis}\,(5))/\{(\alpha, \alpha^{-1}) : \alpha \in \mathbb{C}^*\}$, and that $V(2)$ and $V(5)$ are the unique representations of Heis (2) and Heis (5)-modules respectively with central charge 1. Hence $V(10) = V(2) \otimes V(5)$ as a Heis (10)-module.

Proposition 4.4. *The vector spaces* $\mathrm{Sym}^2(V(2) \otimes V(5))$ *and* $V(2\delta)$ *split as* $\sum_{k=1}^4 I_k$ *and* $\sum_{k=1}^4 J_k$ *respectively, where* I_k *and* J_k *are isotypical components for action of* Heis (δ), *with central charge 2, with* $dim I_k = 15$, $k = 1, 2, 3$, $dim I_4 = 10$ *and* $dim J_k = 10$ *for* $k = 1, 2, 3, 4$. *Moreover,* I_k *and* J_k *correspond to isomorphic representations of* Heis (δ), *for each* k.

Proof. Here Heis (δ) acts on $\mathrm{Sym}^2 V(\delta)$ and $V(2\delta)$, with central charge 2. In fact, Heis (2) and Heis (5) act on $V(4)$ and $V(10)$ respectively with central charge 2. Now any irreducible representation of Heis (δ), with central charge 2, is a tensor product of irreducible representations of Heis (2) and Heis (5), respectively, with central charge 2. Hence we proceed to compute irreducible representations of Heis (2) and Heis (5) with central charge 2.

By 3.2, one deduces Heis (5) has unique irreducible representation, upto scalars, say W, of dimension 5, with central charge 2. Hence $\mathrm{Sym}^2 V(5) \simeq 3W$ and $\bigwedge^2 V(5) \simeq 2W$.

Now, consider the exact sequence,

$$(4)0 \longrightarrow \bigwedge^2 V(2) \otimes \bigwedge^2 V(5) \xrightarrow{i} \mathrm{Sym}^2(V(2) \otimes V(5))$$

$$\xrightarrow{j} \mathrm{Sym}^2 V(2) \otimes \mathrm{Sym}^2 V(5) \longrightarrow 0$$

where

$$i(e_1 \wedge f_1) \otimes (e_2 \wedge f_2) = (e_1 \otimes e_2).(f_1 \otimes f_2) - (e_1 \otimes f_2).(f_1 \otimes e_2),$$

and

$$j(e_1 \otimes e_2).(f_1 \otimes f_2) = e_1 f_1 \otimes e_2 f_2.$$

Also, one easily sees that $\mathrm{Sym}^2 V(2) \simeq V_1 \oplus V_2 \oplus V_3$ for the action of Heis (2), with central charge 2. Since $\mathrm{Sym}^2 V(2) \otimes \mathrm{Sym}^2 V(5) \simeq (V_1 \oplus V_2 \oplus V_3) \otimes 3W$ and $\bigwedge^2 V(2) \otimes \bigwedge^2 V(5) \simeq V_4 \otimes 2W$, $\mathrm{Sym}^2(V(2) \otimes V(5)) = I_1 \oplus I_2 \oplus I_3 \oplus I_4$, where $I_k \simeq 3(V_k \otimes W)$, $k = 1, 2, 3$ and $I_4 \simeq 2(V_4 \otimes W)$ and this is an isotypical decomposition for Heis (δ) with central charge 2. By 3.7, it follows that $V(2\delta) = \sum_{k=1}^{4} J_k$ as a Heis (δ)-module with $\dim J_k = 10$ for each k. $\square$

Remark 4.5. It follows that $\rho_2' = r_1 \oplus r_2 \oplus r_3 \oplus r_4$, where $r_i : I_i \simeq 3(V_i \otimes W) \longrightarrow J_i$, for $i = 1, 2, 3$ and $r_4 : I_4 \simeq 2(V_4 \otimes W) \longrightarrow J_4$. Hence $\mathrm{Ker}\ \rho_2' = \oplus_{i=1}^{4} \mathrm{Ker}\,(r_i)$.

Proposition 4.6. *Suppose r_i is not surjective for some i. Then there are quadrics of rank 4 in the kernel.*

Proof. Notice that in 4.4, $\mathrm{Sym}^2 V(2) = V_1 \oplus V_2 \oplus V_3$ where $V_1 = \mathbb{C}(x_0^2 + x_2^2)$, $V_2 = \mathbb{C}(x_0^2 - x_2^2)$, $V_3 = \mathbb{C}(x_0 \otimes x_1 + x_1 \otimes x_0)$. Hence $I_i = V_i \otimes \mathrm{Sym}^2 V(5)$, for $i = 1, 2, 3$. If r_i is not surjective for some i, $1 \leq i \leq 3$, then $\dim \mathrm{Ker}\,(r_i) \geq 10$. Consider the subvariety $\mathcal{V}_3$ defined as in 4.1, whose points correspond to quadrics of rank less than or equal to 2 in $\mathrm{Sym}^2 V(5)$ and $\dim \mathcal{V}_3 = 8$. Let yz be a quadric of rank ≤ 2 in $\mathrm{Sym}^2 V(5)$. Then $V_i \otimes \mathbb{C}(yz)$ represent quadrics of rank$\leq$ 4 in $\mathrm{Sym}^2 V(10)$. Since $i : P(V_i \otimes \mathrm{Sym}^2 V(5)) \longrightarrow P(\mathrm{Sym}^2 V(5))$ is the natural isomorphism, $\mathcal{V}_3$ intersects the isomorphic image of $P(Ker(r_1))$ in $P(\mathrm{Sym}^2 V(5))$. Call this intersection Q. Then points in $i^{-1}(Q)$ corresponds to quadrics of rank than or equal to 4 in $P(\mathrm{Sym}^2 V(10))$. This gives the assertion when $i = 1, 2, 3$.

Now if r_4 is not surjective, then Ker (r_4) is a vector subspace of dimension at least 5 of I_4.

Let $R = P(\bigwedge^2 V(2) \otimes \bigwedge^2 V(5))$. Then R is a projective space of dimension 9. Fix a basis $\{e_1, f_1\}$ of $V(2)$. Consider the embedding, $G(2, V(5)) \xrightarrow{a} R$, which sends a 2-plane $\mathbb{C}\{e_2, f_2\}$ to $(e_1 \wedge f_1) \otimes (e_2 \wedge f_2)$. Then $G = Im(a)$ is a subvariety of R of dimension 6. >From the exact sequence (4) in 4.4, we have

1) Linear embedding $R \xrightarrow{i'} P(\mathrm{Sym}^2((V(2) \otimes V(5))$ and $i'(R) = P(I_4)$.
2) Points of $i'G$ represent quadrics of rank≤ 4 in $\mathrm{Sym}^2(V(2) \otimes V(5))$.

Hence intersection of $P(\text{Ker }(r_4))$ and $i'G$ in $i'R$ is a subvariety of dimension at least 1, whose points correspond to quadrics of rank $\leq$4 in $P(H^0(L)\check{)}$ containing $\phi_L(A)$. $\square$

If ρ_2 is not surjective, then by 3.7 and 4.6, there are quadrics of rank less than or equal to 4 in the kernel of ρ_2 which contradicts 4.2.

We now consider the cases when $\delta = (1, 8)$ and $\delta = (1, 12)$.

Here $K(\delta)_2 \subset 2K(\delta)$ and so from 3.5, it follows that $\mathrm{Sym}^2 V(\delta) = \sum_{\chi \in \widehat{(K(\delta)_2)}} S_\chi$, where S_χ are the isotypical components as a Heis (δ) module with central charge 2 and

Case 1: When $\delta = (1, 8)$, dim$S_\chi = 12$, if χ is trivial and dim$S_\chi = 8$, if χ is nontrivial.

Case 2: When $\delta = (1, 12)$, dim$S_\chi = 24$, if χ is trivial and dim$S_\chi = 18$, if χ is nontrivial.

Proposition 4.7. *Suppose χ is trivial.*

a) In Case 1, any vector subspace W of S_1 of dimension at least 8, has quadrics of rank less than or equal to 4.

b) In Case 2, any vector subspace W of S_1 of dimension at least 18, has quadrics of rank less than or equal to 4.

Proof. From 3.5, $V(\delta) = \sum_{\chi \in \widehat{K(\delta)_2} W_\chi}$, and $S_1 = \sum_{\chi \in \widehat{K(\delta)_2}} \mathrm{Sym}^2 W_\chi$, where $W_\chi = \{f \in V(\delta) \otimes V(\delta) : s(x)f = \chi(x)f \text{ for } x \in K(\delta)_2\}$. Let W be a vector subspace of S_χ and $R_{\chi,\chi'} = \mathrm{Sym}^2 W_\chi + \mathrm{Sym}^2 W_{\chi'}$, for $\chi \neq \chi'$.

a) Notice that dim$W_\chi = 2$ hence dim $R_{\chi,\chi'} = 6$ and dim $S_1 = 12$. If dim$W \geq 8$, then W intersects any of the subspaces $R_{\chi,\chi'}$ in S_1. Since the elements of $R_{\chi,\chi'}$ correspond to quadrics of rank 4, a) in 4.7 is proved.

b) Here dim$W_\chi = 3$ hence dim $R_{\chi,\chi'} = 12$ and dim $S_\chi = 24$. If dim$W \geq 18$, then $T_{\chi,\chi'} = W \cap R_{\chi,\chi'}$ is of dimension at least 6. But $T_{\chi,\chi'} \subset R_{\chi,\chi'} \subset \mathrm{Sym}^2(W_\chi + W_{\chi'})$. By 4.1, $\mathcal{V}_2 \subset P(\mathrm{Sym}^2(W_\chi + W_{\chi'}))$ parametrizes quadrics of rank less than or equal to 4 and is of dimension equal to 17. Hence $\mathcal{V}_2$ intersects $P(T_{\chi,\chi'})$ and this proves b). $\square$

Proposition 4.8. *Suppose χ is nontrivial.*

a) In Case 1, any vector subspace W of S_χ of dimension at least 4, has quadrics of rank less than or equal to 4.

b) In Case 2, any vector subspace W of S_χ of dimension at least 12, has quadrics of rank less than or equal to 4.

Proof. Notice that $S_\chi = W_{\chi_1}.W_{\chi_2} + W_{\chi_3}.W_{\chi_4}$, such that $\chi_1.\chi_2 = \chi_3.\chi_4 = \chi$. Let W denote a vector subspace of S_χ. Consider the subvarieties

$$Q = \{a \otimes b : a \in P(W_{\chi_1}), b \in P(W_{\chi_2})\} \subset P(W_{\chi_1} \otimes W_{\chi_2})$$

and

$$Q' = \{c \otimes d : c \in P(W_{\chi_3}), d \in P(W_{\chi_4})\} \subset P(W_{\chi_3} \otimes W_{\chi_4}).$$

Consider the join of Q and Q', denoted by $Q + Q'$, in $P(S_\chi)$. Then points of $Q + Q'$ correspond to quadrics of rank less than or equal to 4 in $\mathrm{Sym}^2 V(\delta)$.

a) Here $\dim W_\chi.W_{\chi'} = 4$ and $\dim Q + Q'$ is at least 5. Hence $P(W)$ intersects $Q + Q'$ in $P(S_\chi)$. This proves a).

b) Here $\dim W_{\chi_i}.W_{\chi_j} = 9$ and $\dim Q + Q'$ is at least 9. Hence $P(W)$ intersects $Q + Q'$ in $P(S_\chi)$. This proves b). $\square$

Consider the Heis (δ) equivariant morphism $\rho_2' : \mathrm{Sym}^2 V(\delta) \longrightarrow V(2\delta)$. Then ρ_2' restricts to Heis (δ) equivariant morphisms on the four isotypical components of $\mathrm{Sym}^2 V(\delta)$. If the restrictions are not surjective, then by 3.7, 4.7 and 4.8, there are quadrics of rank less than or equal to 4 in the kernel of ρ_2. This contradicts 4.2.

We now show surjectivity of ρ_n, for $n \geq 3$. Consider a generic hyperplane section of $\mathcal{O}_{\mathbb{P}}(1)$ and intersect with A. Then by Bertini's theorem, it is a smooth curve, say C. Let K_C denote the canonical bundle on C. Then deg $K_C = \deg L/C = C^2 = 2d$, by Riemann Roch theorem. Hence $2g_C - 2 = 2d$ gives $g_C = d + 1$. One easily sees that ϕ_L restricted to C, factors as $C \longrightarrow P(V\check{}) \subset P(H^0(L)\check{})$ where $V \subset H^0(C, K_C)$ is a subspace of dimension $d - 1$.

Consider the exact sequence of sheaves on A, for $n \geq 1$,

$$0 \longrightarrow L^{n-1} \longrightarrow L^n \longrightarrow L^n|C \longrightarrow 0$$

When $n = 1$, we get the following exact sequence of vector spaces,

$$0 \longrightarrow H^0(\mathcal{O}_A) \longrightarrow H^0(L) \longrightarrow H^0(K_C) \longrightarrow H^1(\mathcal{O}_A) \longrightarrow 0$$

and $V = H^0(L)/H^0(\mathcal{O}_A)$.

Now consider the following commutative diagram, for $n \geq 2$,

$$\begin{array}{ccccccc} 0 \longrightarrow & I_n & \longrightarrow & \mathrm{Sym}^n\, H^0(L) & \xrightarrow{\rho_n} & H^0(L^n) \\ & \downarrow\gamma & & \downarrow\alpha & & \downarrow\beta \\ 0 \longrightarrow & I'_n & \longrightarrow & \mathrm{Sym}^n\, V & \xrightarrow{\rho'_n} & H^0(C,\ K_C^n) \end{array}$$

We observe that α, β are surjective and dim Ker $\gamma =$ dim I_{n-1}, hence dim $I_n \leq$ dim $I_{n-1} +$ dim I'_n. Consider

$$0 \longrightarrow \mathrm{Ker}\ \eta \longrightarrow I'_n \xrightarrow{\eta} I'_n|_H$$

where H is a generic hyperplane in $P(V\check{})$. Now,

$$\dim\ \mathrm{Ker}\ \eta = \dim\ I'_{n-1}$$

and

$$\begin{aligned} \dim\ I_n{}'|_H &\geq \dim\ I_n{}' - \dim\ I'_{n-1} \\ &\geq \dim\ I_n - \dim\ I_{n-1} - \dim\ I'_{n-1} \end{aligned}$$

Since H is a generic codim 2 plane in $\mathbb{P}$, it intersects the abelian surface at 2d points which can be assumed to be in general position in H. Hence $I'_n|_H$ is a vector space of degree n hypersurfaces in H vanishing on 2d points which are in general position. We prove by induction on n, that ρ_n is surjective.

We have proved above, ρ_2 is surjective. Suppose ρ_{n-1} is surjective, $n \geq 3$. If ρ_n is not surjective, then

$$\dim\ I_n > \binom{d-1+n}{n} - dn^2.$$

We use the general fact:

Fact 4.9. In $\mathbb{P}^r$, $nr+1$ points in general position pose independent condition on the vector space of degree n hypersurfaces.

Here $n \geq 3$, so $n(d-3)+1 > 2d$, except when $n = 3$ and $d = 7$. So the dimension of the vector space of degree n hypersurfaces in H vanishing on $2d$ points is $\binom{d-3+n}{n} - 2d$.

Let $\dim I_n = \binom{d-1+n}{n} - dn^2 + x$ where $x > 0$. Then one easily checks that

$$\begin{aligned}
&\dim I'_n|_H - \binom{n+d-3}{n} + 2d \geq \dim I_n \\
&\quad - \dim I_{n-1} - \dim I'_{n-1} - \binom{n+d-3}{n} + 2d \\
&= \binom{d-1+n}{n} - dn^2 + x - \binom{d-2+n}{n-1} + d(n-1)^2 \\
&\quad - \binom{d-3+n}{n-1} + (2n-3)d - \binom{n+d-3}{n} + 2d. \\
&= \binom{d-1+n}{n} - \binom{d-2+n}{n-1} - \binom{d-3+n}{n-1} - \binom{n+d-3}{n} + x \\
&= x.
\end{aligned}$$

Hence $\dim I'_n|_H > \binom{d-3+n}{n} - 2d$, contradicting above stated fact. When $n = 3$ and $d = 7$, similar computation shows $\dim I'_n|_H - \binom{n+d-3}{n} + n(d-3) + 1 \geq x - 1$. But by 3.1, $x \geq 7$. Hence this contradicts 4.9.

This completes the proof of the theorem. $\square$

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References

[1] Adler, A., Ramanan, S.: *Moduli of Abelian Varieties*. Berlin–Heidelberg–New York: Springer-Verlag, 1996

[2] Birkenhake, Ch., Lange, H.: *Complex abelian varieties*. Berlin–Heidelberg–New York: Springer-Verlag, Berlin, 1992

[3] Harris, J.: *Algebraic Geometry: A First Course*. Berlin–Heidelberg–New York: Springer-Verlag, 1992

[4] Lazarsfeld, R.: *Projectivite normale des surface abeliennes*. Redige par O. Debarre. Prepublication No. **14**, Europroj- C.I.M.P.A., Nice, (1990)

[5] Mumford, D.: On the equations defining Abelian varieties 1. Invent. math. **1**, 287–354 (1966)

[6] Mumford, D., Nori, M., Norman, P.: *Tata Lectures on Theta III*. Basel–Boston: Birkhauser, 1991

[7] Pirola, G.P.: Base Number Theorem for Abelian Varieties. Math. Ann. **282**, 361–368 (1988)