Handlebody Decomposition of a Manifold

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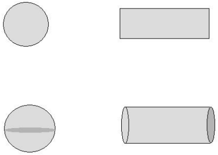
Introduction

A manifold M of dimension n can be viewed as a subspace of some $\mathbb{R}^N$ which is locally diffeomorphic to open subsets of $\mathbb{R}^n$. For example, we can consider the spheres of all dimensions, Torus, a surface of any genus, cylinders etc. Given a manifold we want to determine its algebraic invariants. The Algebraic tools in hand are the homology and cohomology theories. However, it is not always an easy task to compute these invariants. An idea is to decompose a manifold into simple objects in a useful way so as to conclude something about its homology groups.

What is a handle?

A *handle* of index k and dimension n is a manifold with boundary which is diffeomorphic to $D^k \times D^{n-k}$ in $\mathbb{R}^n$, where D^k and D^{n-k} denote balls in Euclidean spaces $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ respectively.

In dimension 2 A handle of index 0 or 2 is a disc and a handle of index 1 is a rectangle.



In dimension 3: A handle of index 0 or 3 is a 3-ball and a handle of index 1 or 2 is a cylinder.

Attaching of a handle

Let M_0 be an n-dimensional manifold with boundary and let

$$\phi : S^{k-1} \times D^{n-k} \to M_0$$

be a smooth embedding such that image ϕ is contained in ∂M_0.
Define an equivalence relation on the disjoint union

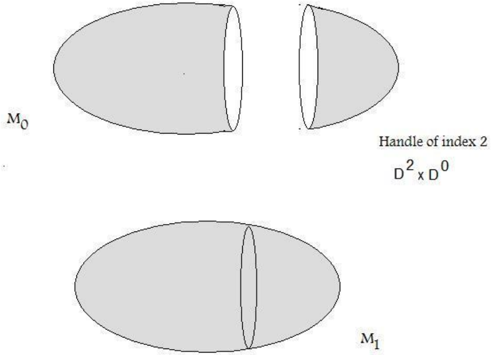
$$M_0 \coprod D^k \times D^{n-k}$$

by

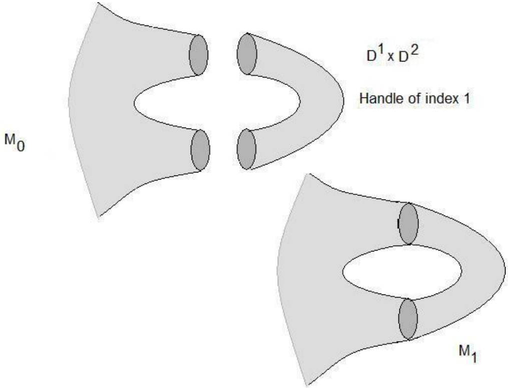
$$x \sim \phi(x) \quad \text{for each} \quad x \in S^{k-1} \times D^{n-k}.$$

The quotient space M_1 is said to be obtained from M_0 by attaching a handle of index k.
M_1 need not be a smooth manifold.

Attaching of a handle of index 2 in dimension 2.



Attaching of a handle of index 1 in dimension 3.



Smooth attaching

Take a small closed neighbourhood T of $S^{k-1} \times D^{n-k}$ and an embedding $\psi : T \to M_0$ such that $\psi(\partial T \setminus S^{k-1} \times \{0\}) \subset \partial M_0$.
The quotient space

$$M_1 = M_0 \coprod D^k \times D^{n-k} / \sim$$

is a smooth manifold possibly with boundary.
The homotopy type of M_1 is different from that of M_0.

Main Result

The following is a fundamental result in Morse Theory.

Theorem

Let M be a compact manifold without boundary. Then M can be developed from a ball D^n by successively attaching to it finitely many handles of dimension n.

Non-degenerate critical points

Let M be a manifold and $f : M \to \mathbb{R}$ a smooth real-valued function on M. A point $p \in M$ is said to be a *critical point* of f if the derivative of f at p is zero.
A critical point p on M is called a *non-degenerate* critical point if the Hessian of f at p is non-singular. Hence, if p is a non-degenerate critical point of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ then

$$\nabla f(p) \equiv (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})(p) = 0$$

$$H_f(p) \equiv \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j} \quad \text{is non-singular}$$

The number of negative eigenvalues of the Hessian $H_f(p)$ is called the *index* of the critical point p.

Morse Lemma

Examples. Let $M = \mathbb{R}^3$. Consider quadratic polynomials

$$f(x,y,z) = x^2 + y^2 + z^2,$$
$$g(x,y,z) = x^2 + y^2 - z^2,$$
$$h(x,y,z) = x^2 - y^2 - z^2.$$

Note that $(0,0,0)$ is the only critical point of f, g and h. Moreover, it is a non-degenerate critical point of each of these functions.

Morse Lemma. *If p is a non-degenerate critical point of f of index λ, then there exists a coordinate system $(x_1, x_2, \ldots, x_n)$ around p such that*

$$f(x_1, x_2, \ldots, x_n) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 - \cdots - x_n^2.$$

Therefore, it is enough to consider quadratic polynomials in order to study the local behaviour of a function near a non-degenerate critical point.

Critical point versus topology of level sets

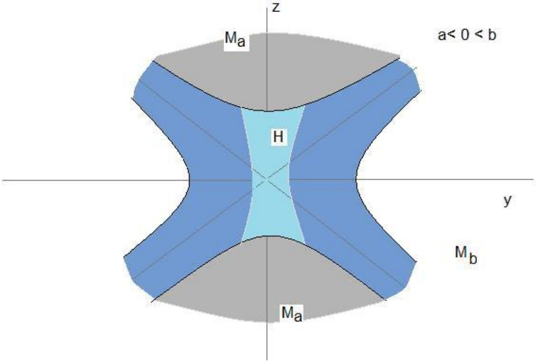
Let $f : M \to \mathbb{R}$ be a smooth function. For any real number c define

$$M^c = \{x \in M | f(x) \leq c\}.$$

Suppose that p is the only non-degenerate critical point of f.
The value of f at p, namely $f(p)$ will be called a *critical value* of f. Let a and b be two real numbers such that $a < f(p) < b$.

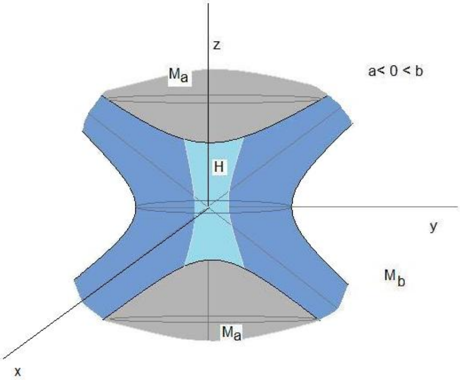
- The topological type of M^a is the same for all $a < f(p)$.
- The topological type of M^a is the same for all $b > f(p)$.
- The topological structure of M^b is different from that of M^a.

Consider the polynomial $f = y^2 - z^2$ on $\mathbb{R}^2$. $(0, 0)$ is the only critical point of f and $f(0, 0) = 0$.



M_b has the same homotopy type as M_a attached with the handle H. Index of the critical point = index of the handle H.

Let $g(x, y, z) = x^2 + y^2 - z^2$ and $a < 0 < b$.



M_b has the same homotopy type as M_a attached with the handle H.
Index of the critical point = index of the handle H.

For any real numbers $a, b > a$, define the sets $M^{[a,b]}$ as follows:

$$M^{[a,b]} = \{x \in M | a \leq f(x) \leq b\}$$

Theorem

If $M^{[a,b]}$ is compact and contains no critical point of $f : M \to \mathbb{R}$ then M^a is diffeomorphic to M^b.

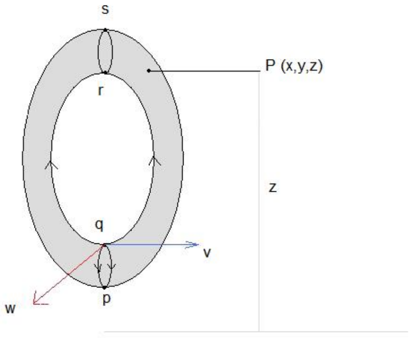
Theorem

Suppose that $M^{[a,b]}$ is compact and contains exactly one non-degenrate critical point of f. If the index of the critical point is k then M^b is obtained from M^a by successively attaching a handle of index k and a collar.

Height function h on the vertical torus

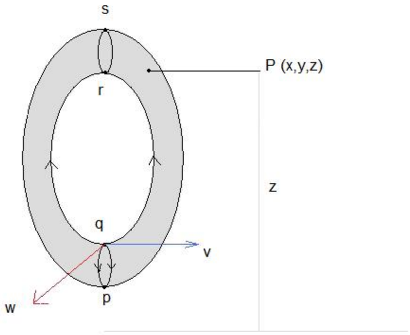
Let $\mathbb{T}^2$ be the vertical torus. Consider the height function $h : \mathbb{T}^2 \to \mathbb{R}$ given by

$$h(x, y, z) = z, \quad \text{for} \quad (x, y, z) \in \mathbb{T}^2.$$



h has exactly four critical points namely at p, q, r and s.

h is a Morse function on $\mathbb{T}^2$.



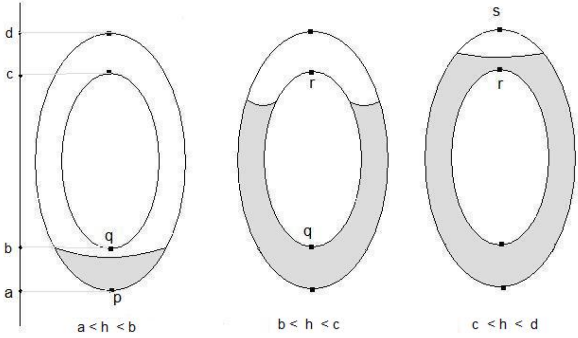
- p is a minima and s is a maxima. The index of h at p is 0 and the index of f at s is 2.
- The points q and r are saddle points. The indices of f at q and r are both equal to 1.

The level sets of h

Recall that critical points of h are p, q, r and s.
Let $h(p) = a, \quad h(q) = b, \quad h(r) = c, \quad h(s) = d$.

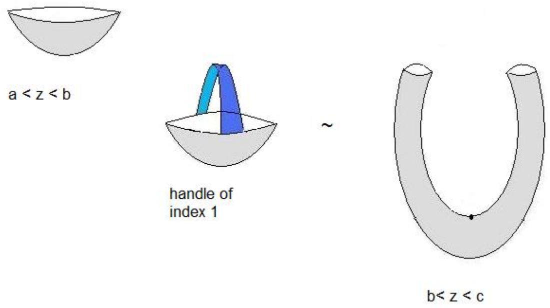
- $a < z < b \Rightarrow M^z$ is diffeomorphic to a disc; has one boundary component diffeomorphic to circle.
- If $b < z < c \Rightarrow M^z$ is diffeomorphic to a cylinder; has two boundary components each diffeomorphic to a circle.
- If $c < z < d$, M^z is diffeomorphic to $\mathbb{T}^2$ minus an open disc; has only one boundary component that is diffeorphic with a circle.
- If $z \geq d$ then M^z is the complete torus which is without boundary.
- None of M^b and M^c is a manifold.

Recall that critical points of h are p, q, r and s, and $h(p) = a, \quad h(q) = b, \quad h(r) = c, \quad h(s) = d$.



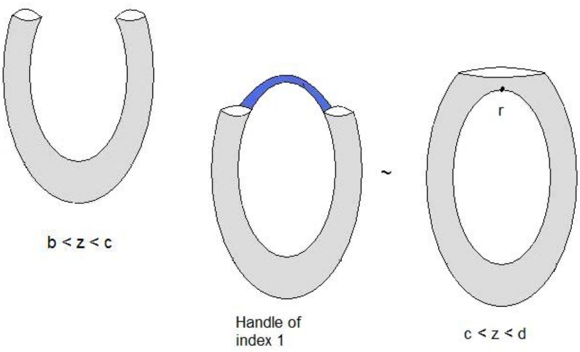
Level Sets M^z of h.

We can build up the torus from a disc by successively attaching a handle for each critical point q, r and s.
The critical point q has index 1. The critical value at q is b.



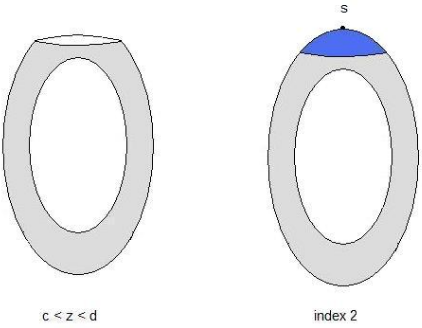
Attaching of handle of index 1

The critical point r has index 1. The critical value at r is c.



Attaching of handle of index 1

The critical point s has index 2. The critical value at s is d.



Attaching of handle of index 2

Global theorems

A smooth function $f : M \longrightarrow \mathbb{R}$ is said to be a Morse function if all its critical points are non-degenerate.

Proposition

Every smooth manifold admits a Morse function $f : M \longrightarrow \mathbb{R}$ such that M^a is compact for every $a \in \mathbb{R}$.

Theorem

Let $f : M \longrightarrow \mathbb{R}$ be a Morse function on a compact manifold M such that each M^a is compact. Let

$$p_1, p_2, \ldots, p_k, \ldots$$

be the critical points of f and let index of p_i be λ_i.
Then M has a handlebody decomposition which admits a handle of index λ_i for each i.

Proof

Take a Morse function $f : M \to \mathbb{R}$.
Let $c_1 < c_2 < \cdots < c_k < \ldots$ be the critical values of f.
Choose real numbers $\{a_i\}$ satisfying $c_i < a_i < c_{i+1}$.
For any real numbers $a, b > a$, define the sets $M^{[a,b]}$ as follows:

$$M^{[a,b]} = \{x \in M | a \leq f(x) \leq b\}$$

Then $M^{a_{i+1}} = M^{a_i} \cup M^{[a_i, a_{i+1}]}$ and

$$M = \cup_{i=0}^{m-1} M^{[a_i, a_{i+1}]}.$$

Since $M^{[a_i, a_{i+1}]}$ is compact it can contain only finitely many critical points of f. Therefore, it has the homotopy type of a handle of index λ_{i+1}, where λ_{i+1} is the index of c_{i+1}.
This completes the proof.

Corollary

Every smooth manifold has the homotopy type of a CW complex.

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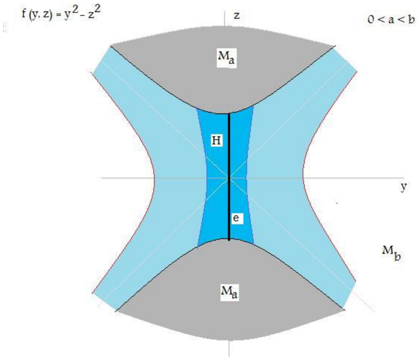
Proof:



M_b is homotopically equivalent to M_a attached with a cell of dimension 1.

Betti Numbers and Euler Characteristic

Let M be a any topological space. For each integer $k \geq 0$ we can associate a group $H_q(M; \mathbb{Z})$, called the q-th homology group of M. If M and N are homotopically equivalent, then $H_q(M, \mathbb{Z})$ is isomorphic to $H_q(N, \mathbb{Z})$. Further, if $M_0 \subset M_1$ then we can defined relative homology groups $H_q(M_1, M_0)$ which are isomorphic to $H_q(M_1/M_0)$ for $q > 0$. The integer $\beta_q(M) = \text{ rank } H_q(M; \mathbb{Z})$ is called the q-th Betti number of M. The Euler characteristic of M is defined as

$$\chi(M) = \sum_k (-1)^k \beta_k.$$

These are important algebraic invariants of topological spaces.

Morse Inequalities

Theorem

Let $f : M \to \mathbb{R}$ be a Morse function on a compact manifold M. Let C_k denote the number of critical points of f of index k. Then we have the following relations:

$$\beta_k \leq C_k, \quad \text{for each } k,$$
$$\chi(M) = \sum_k (-1)^k C_k.$$

Proof:

First observe that if M_1 is obtained from M_0 by attaching a handle of index k, then $H_q(M_1, M_0) = H_q(D^k, S^{k-1})$. Hence,

$$H_q(M_1, M_0) = \mathbb{Z} \text{ if } q = k \text{ and } 0 \text{ otherwise.}$$

Let $c_1 < c_2 < \cdots < c_k$ be the critical values of f.
Choose real numbers $\{a_i\}$ such that $c_i < a_i < c_{i+1}$, $0 \leq i \leq k-1$, and let $M_i = M^{a_i}$. We can therefore get an increasing sequence of manifolds with boundary:

$$D^n = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = M,$$

where M_{i+1} is obtained from M_i by attaching a finitely many cells, one for each critical point in $M^{[a_i, a_{i+1}]}$. Since H_q is sub-additive we get

$$\operatorname{rank} H_q(M; \mathbb{Z}) \leq \sum_i \operatorname{rank} H_q(M_{i+1}, M_i).$$

Thus $\operatorname{rank} H_q(M; \mathbb{Z})$ is less than or equal to the number of critical points of index q.

Reference

John Milnor: *Morse Theory*. Annals of Mathematics Studies. Princeton University Press.